

Structural Estimation of Dynamic Macroeconomic Models using Higher-Frequency Financial Data*

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Abstract

In this paper we show how high-frequency financial data can be used in a combined macro-finance framework to estimate the underlying structural parameters. Our formulation of the model allows for substituting macro variables by asset prices in a way that enables casting the relevant estimation equations partly (or completely) in terms of financial data. We show that using only financial data allows for identification of the majority of the relevant parameters. Adding macro data allows for identification of all parameters. In our simulation study, we find that it also improves the accuracy of the parameter estimates. In the empirical application we use interest rate, macro, and S&P500 stock index data, and compare the results using different combinations of macro and financial variables.

JEL classification: C13; E32; O40

Keywords: Structural estimation; High-Frequency financial data; AK-Vasicek model; Martingale Estimation Function; Generalized Method of Moments

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1 Introduction

One important lesson from the financial crisis of 2007/2008 is the need for a joint framework, which overcomes the traditional separation of macroeconomic and finance. Over the last decade, a large literature developing models at the intersection of macroeconomics and finance has emerged (among others Rudebusch and Swanson 2012, Gürkaynak and Wright 2012). Most of the papers, however, focus on the interaction of macro variables, fiscal and monetary policy, and their implications for the term structure of interest rates. In a macro-finance framework, the asset pricing kernel is consistent with the macroeconomic dynamics. So the open question is to what extent financial data can be used to replace macroeconomic variables in structural estimation.

In this paper we exploit asset pricing implications of a simple macro-finance model to cast the relevant estimation equations partly (or completely) in terms of financial data. This allows us to estimate the structural parameters using only financial data, only macroeconomic data, or a combination of these. Our motivation for doing so is that macro data, in contrast to financial data, are usually available at lower frequencies and subject to substantial revisions. Given the relatively high volatility of financial data, we investigate the informational content of various financial data including interest rates, bond and stock prices for the dynamics of macroeconomic aggregates. Our aim is to provide new insights into the use of financial data in a simple macro-finance model, and to derive implications for the estimation of more elaborated models.

The structure of our approach is as follows. We start by describing the model, similar to Christensen, Posch and van der Wel (2016), and derive the stochastic discount factor (SDF), i.e., the asset pricing kernel which allows to price any financial asset in the economy. In a second step we define various financial variables, compute their price dynamics, and cast the model's equilibrium dynamics in terms of financial data alone or combined with macro data, and the structural parameters. In a third step, we estimate the structural parameters of the model using the Generalized Method of Moments (GMM) and Martingale Estimation Function (MEF), with different specifications and different types of financial data. We study the identification of parameters both in a simulation study and empirically using interest rate, macro, and S&P500 stock index data.

Our results obtained from both simulation study and empirical estimation indicate that using a combined macro-finance framework not only improves the identification of structural parameters but also the accuracy of the estimates. Another important feature of our macro-finance estimation approach is that it drastically reduces the upward-bias typically encountered in similar mean-reverting interest rate models in the literature.

The rest of the paper is structured as follows. Section 2 describes the used macro framework and derives the general equilibrium price for the claim on future dividends. The following section

is devoted to the derivation of the systems of equilibrium equations and the derivation of the different estimators. Furthermore, the interdependencies between macro and finance dynamics as suggested by the model are discussed. Before turning to the empirical estimation in section 5, we run various simulation studies in section 4 to evaluate small sample properties and to test identification and different parameter restrictions.

2 The Macro Framework and Asset-Pricing

2.1 The Macro Framework

In this paper we use the framework of the continuous time stochastic AK-model by Christensen, Posch & van der Wel (2016). We start from this simple framework because we want to keep it as simply as possible while maintaining the capability of explaining both macroeconomic and financial dynamics. Another important benefit of this approach is the availability of analytical solutions that offer an intuitive and consistent way to replace macro with financial variables. We will only summarize the main properties of the macro-model and refer to Christensen, Posch & van der Wel (2016) for a more detailed overview.

At each instance in time, output Y_t is generated by combining capital, factor productivity and a constant amount of labor

$$Y_t = A_t F(K_t, L)$$

Here the aggregate capital stock is given by K_t , total factor productivity (TFP) is represented by A_t while L is the constant population size. In this economy TFP is driven by B_t , a standard Brownian motion, with $\mu(A_t)$ representing the generic drift- and $\eta(A_t)$ the generic volatility function.

$$dA_t = \mu(A_t)dt + \eta(A_t)dB_t$$

If gross investments, I_t , are higher than capital depreciation, K_t increases according to

$$dK_t = (I_t - \delta K_t)dt + \sigma K_t dZ_t$$

with σ being the volatility of stochastic depreciation, δ representing the depreciation rate and Z_t being another standard Brownian motion.

The equilibrium conditions in this economy are standard: production factors are rewarded with their marginal products $r_t = Y_K$ and $w_t = Y_L$. The good market clearing condition is given by $Y_t = C_t + I_t$. Households in this economy are represented by a representative household. This stand-in consumer exhibits additively separable utility and maximizes expected life time utility. The Euler-equation is given by

$$\begin{aligned} \frac{dU_C}{U_C} &= (\rho - (r_t - \delta))dt - \frac{U_{CC}(C_t, A_t)}{U_C(C_t, A_t)} C_K \sigma^2 K_t dt + \frac{U_{CC}(C_t, A_t)}{U_C(C_t, A_t)} C_A \eta(A_t) dB_t \\ &+ \frac{U_{CA}(C_t, A_t)}{U_C(C_t, A_t)} \eta(A_t) dB_t + \frac{U_{CC}(C_t, A_t)}{U_C(C_t, A_t)} C_K \sigma K_t dZ_t \end{aligned}$$

We are again following the model by Christensen, Posch & van der Wel (2016), and use the mean-reversion Vasicek specification for the rental rate on physical capital, r_t . We denote the speed of mean-reversion by κ and the long-term mean of the interest rate by γ . Despite its simplicity the Vasicek interest rate model still plays a crucial role in the finance literature and hence offers a reasonable interest rate process. The equilibrium dynamics of the economy are then given by

$$d \ln C_t = (r_t - \rho - \delta - \frac{1}{2}\sigma^2)dt + \sigma dZ_t \quad (1a)$$

$$d \ln K_t = (r_t - \rho - \delta - \frac{1}{2}\sigma^2)dt + \sigma dZ_t \quad (1b)$$

$$d \ln Y_t = (\mu(r_t)/r_t + r_t - \rho - \delta - \frac{1}{2}\eta(r_t)^2/r_t^2 - \frac{1}{2}\sigma^2)dt + \eta(r_t)/r_t dB_t + \sigma dZ_t \quad (1c)$$

$$dr_t = \kappa(\gamma - r_t)dt + \eta dB_t \quad (1d)$$

The above system is completely cast in terms of macroeconomic variables. Our aim is the estimation of the six structural parameters of the model that are given by the vector

$$\phi = (\kappa, \gamma, \eta, \rho, \delta, \sigma)^T \quad (2)$$

In our estimation we are using different combinations of financial and macro data to cast the relevant estimation equations partly and completely in terms of financial data.

2.2 The Stochastic Discount Factor

In order to obtain the asset pricing implications of the model we now derive the stochastic discount factor. Following Hansen and Scheinkman(2009),the SDF for $s > t$ can be obtained from the Euler equation as the process

$$\frac{\Lambda_s}{\Lambda_t} = e^{-\rho(s-t)} \frac{V_K(K_s, A_s)}{V_K(K_t, A_t)} \quad (3)$$

For $U(C_t, A_t) = U(C_t)$ and by using the analytical solution $C_t = \rho K_t$

$$\frac{dU_c}{U_c} = (\rho - (r_t - \delta))dt - \frac{U_{cc}(C_t)}{U_c(C_t)} C_K \sigma^2 K_t dt + \frac{U_{cc}(C_t)}{U_c(C_t)} C_K \sigma K_t dZ_t$$

Hence, the SDF reads

$$d\Lambda_t = (-(r_t - \delta))\Lambda_t dt - \frac{U_{cc}(C_t)}{U_c(C_t)} C_K \sigma^2 K_t \Lambda_t dt + \frac{U_{cc}(C_t)}{U_c(C_t)} C_K \sigma K_t \Lambda_t dZ_t \quad (4)$$

We can derive the certainty equivalent rate of return by

$$-\frac{1}{dt} E_t \left[\frac{d\Lambda_t}{\Lambda_t} \right] = r_t - \delta + \frac{U_{cc}(C_t)}{U_c(C_t)} C_K \sigma^2 K_t \equiv r_t^f \quad (5)$$

Applying the AK-specification $U(C_t) = \ln(C_t)$ and $C_t = \rho K_t$ to the evolution of Λ_t we obtain

$$d\Lambda_t = (-(r_t - \delta) + \sigma^2)\Lambda_t dt - \sigma\Lambda_t dZ_t \quad (6)$$

In the appendix we show how to apply Ito's formula to obtain the stochastic discount factor as the process:

$$\frac{\Lambda_s}{\Lambda_t} = e^{-\int_t^s (r_v - \delta - \frac{1}{2}\sigma^2)dv - \sigma \int_t^s dZ_v} \quad (7)$$

2.3 Asset Pricing

We define the stochastic discount factor and use it together with equations (8) and (9) to price the below defined financial assets. Since we are interested in equilibrium prices the SDF allows us to price any asset in this economy consistently with macro dynamics. Note that we assume that there are markets for contingent claims that are all in zero-supply in equilibrium. To find the equilibrium prices we compute (see e.g. Cochrane, 2005)

$$P_t = E_t \left[\frac{\Lambda_s}{\Lambda_t} X_s \right] \quad (8)$$

that is for $s > t$ the pricing equation states that the equilibrium price P of an asset at time t is given by the conditional expectation of the product of the stochastic discount factor and the future payoff X_s . Furthermore we obtain equilibrium returns by using (e.g. Cochrane, 2005)

$$R_s = \frac{X_s}{P_t} \quad (9)$$

2.3.1 Using Interest Rate Data

2.3.1.1 Short Rate

Since the Vasicek specification for the rental rate is an Ornstein-Uhlenbeck process, we show in the appendix how this allows us to find a solution for $s > t$ by using a standard technique for differential equations.

$$r_s = e^{-\kappa(s-t)}r_t + (1 - e^{-\kappa(s-t)})\gamma + \eta e^{-\kappa(s-t)} \int_t^s e^{\kappa(u-t)} dB_u \quad (10)$$

Since the capital stock is unobservable we remove its redundant equation from system (1a) and obtain:

$$d \ln C_t = (r_t - \rho - \delta - \frac{1}{2}\sigma^2)dt + \sigma dZ_t \quad (11a)$$

$$d \ln Y_t = (\mu(r_t)/r_t + r_t - \rho - \delta - \frac{1}{2}\eta(r_t)^2/r_t^2 - \frac{1}{2}\sigma^2)dt + \eta(r_t)/r_t dB_t + \sigma dZ_t \quad (11b)$$

$$dr_t = \kappa(\gamma - r_t)dt + \eta dB_t \quad (11c)$$

Nevertheless, the rental rate on physical capital is not a directly observable variable. One way to deal with the resulting difficulties in estimating system (11a) is using a latent variable approach. Furthermore, we can use the model specification to derive an alternative expression for the rental rate in terms of observable variables and model parameters. In the next section we show how the return on a risk-free bond can be used as a substitute for the rental rate.

2.3.1.2 Risk-Free Bond

Starting from we equation (5) we can define the certainty equivalent rate of return, r_t^f , using the stochastic discount factor as:

$$r_t^f = r_t - \delta - \sigma^2 \quad (12)$$

Note the price of an asset which is paying continuously at the risk-free rate is given by:

$$\begin{aligned} P_{f,t} &= E_t \left[\frac{\Lambda_s}{\Lambda_t} e^{\int_t^s r_v^f dv} \right] = E_t \left[\frac{\Lambda_s}{\Lambda_t} e^{\int_t^s r_v - \delta - \sigma^2 dv} \right] \\ &= E_t \left[e^{-\int_t^s (r_v - \delta - \frac{1}{2}\sigma^2) dv - \sigma \int_t^s dZ_v} e^{\int_t^s (r_v - \delta - \sigma^2) ds} \right] \\ \ln(P_{f,t}) &= 0 \\ P_{f,t} &= 1 \end{aligned}$$

And the return of this asset is given by:

$$R_{f,s} = e^{\int_t^s r_v^f dv} = e^{\int_t^s r_v - \delta - \sigma^2 dv}$$

As shown in appendix (A.2) the instantaneous return on the risk-free asset is given by

$$d \ln R_{f,t} = (r_t - \delta - \sigma^2) dt \quad (13)$$

As expected, this is in line with the above defined risk-free rate. To obtain an expression for the risk-free rate for period $s > t$ we use the Vasicek specification.

$$dr_t^f = \kappa(\gamma - r_t^f - \delta - \sigma^2) dt + \eta dB_t$$

Again, this is an Ornstein Uhlenbeck process whose solution can be obtain in same manner as before:

$$r_s^f = e^{-\kappa(s-t)} r_t^f + (1 - e^{-\kappa(s-t)}) (\gamma - \delta - \sigma^2) + \eta e^{-\kappa(s-t)} \int_t^s e^{\kappa(u-t)} dB_u \quad (14)$$

Replacing the unobservable rental rate on physical capital by the risk-free rate, we can estimate a macro-finance version of the model, by casting the model in terms of :

$$dr_t = \kappa(\gamma - r_t) dt + \eta dB_t, \quad \text{where} \quad r_t = r_t^f + \delta + \sigma^2 \quad (15a)$$

2.3.2 Using Stock Market Data

2.3.2.1 Claim on Future Dividends

Starting from (8) consider a claim on all future dividends (in an endowment economy this is equivalent to a claim on the tree, *not* only on the next period's fruit), that is we have the price

$$P_{d,t} = E_t \left[\int_t^\infty \frac{\Lambda_s}{\Lambda_t} Y_s ds \right] \quad (16)$$

To find the equilibrium price of this asset we have to compute an expression for Y_s , the output in period s . Note that in our model specification we have that $Y_t = A_t K_t$, implying that $Y_s = r_s K_s$. Note that we have already defined r_s in (7). In appendix (A.4) we show how to use the model properties together with Ito's formula to obtain an expression for the capital stock in period $s > t$

$$K_s = K_t e^{\int_t^s (r_v - \rho - \delta - \frac{1}{2}\sigma^2) dv + \sigma \int_t^s dZ_v} \quad (17)$$

Note that we can cast Y_s analogously to (15) in Christensen, Posch & van der Wel (2016) and obtain (see appendix)

$$Y_s = Y_t e^{\kappa\gamma \int_t^s \frac{1}{r_v} dv - \frac{\eta^2}{2} \int_t^s \frac{1}{r_v^2} dv + \int_t^s (r_v - \delta - \rho - \kappa - \frac{1}{2}\sigma^2) dv + \sigma \int_t^s dZ_v + \int_t^s \frac{\eta}{r_v} dB_v} \quad (18)$$

However, here we use a slightly different formulation using (7) together with (17). Hence, multiplying the two equations we arrive at

$$Y_s = \left[K_t r_t + K_t \gamma e^{\kappa(s-t)} - K_t \gamma + K_t \eta \int_t^s e^{\kappa(u-t)} dB_u \right] e^{\int_t^s (r_v - \delta - \rho - \kappa - \frac{1}{2}\sigma^2) dv + \sigma \int_t^s dZ_v} \quad (19)$$

To find the price of the claim on future dividends we use (16) together with (7) and (19) and obtain

$$\begin{aligned} P_{d,t} &= E_t \left[\int_t^\infty \left[K_t r_t + K_t \gamma e^{\kappa(s-t)} - K_t \gamma + K_t \eta \int_t^s e^{\kappa(u-t)} dB_u \right] e^{-\int_t^s (\rho + \kappa) dv} ds \right] \\ &= E_t \left[\int_t^\infty \left[K_t r_t + K_t \gamma e^{\kappa(s-t)} - K_t \gamma + K_t \eta \int_t^s e^{\kappa(u-t)} dB_u \right] e^{-(\rho + \kappa)(s-t)} ds \right] \\ &= \int_t^\infty [K_t r_t - K_t \gamma] e^{-(\rho + \kappa)(s-t)} ds + \int_t^\infty K_t \gamma e^{-\rho(s-t)} ds \end{aligned}$$

Solving the integrals yields we arrive at

$$P_{d,t} = K_t \left[\frac{r_t - \gamma}{(\rho + \kappa)} + \frac{\gamma}{\rho} \right] \quad (20)$$

This is an intuitive result. The price of the claim is based on the sum of two annuities. Recall that in the AK-Vasicek model the parameter γ can be interpreted as the long-term mean of the interest rate, or, since $A_t = r_t$, the long-term mean of total factor productivity. Therefore,

the price of the claim is based on the current capital stock times γ plus the current capital stock times the current level of A_t minus γ . Since this term can be either positive, zero or negative, the price of the claim raises when current total factor productivity A_t is higher than its long-term level and decreases if the current level of technology lies below its equilibrium level.

As shown in the appendix we can take the derivative of the equilibrium price equation (2.3.3) and obtain

$$dP_{d,t} = P_{d,t}[r_t - \rho - \delta]dt + P_{d,t}\sigma dZ_t + P_{d,t}\frac{[\rho\kappa(\gamma - r_t)]}{\rho r_t + \kappa\gamma}dt + P_{d,t}\frac{\rho\eta}{\rho r_t + \kappa\gamma}dB_t \quad (21)$$

or applying Ito's formula to find an expression for the log price change of the claim

$$\begin{aligned} d\ln P_{d,t} &= \left[r_t - \delta - \rho - \frac{1}{2}\sigma^2 + \frac{\rho\kappa(\gamma - r_t)}{[\rho r_t + \kappa\gamma]} - \frac{1}{2}\frac{(\rho\eta)^2}{[\rho r_t + \kappa\gamma]^2} \right] dt \\ &\quad + \frac{\rho\eta}{[\rho r_t + \kappa\gamma]}dB_t + \sigma dZ_t \end{aligned} \quad (22)$$

or

$$d\ln P_{d,t} = d\ln C_t + \left[\frac{\rho\kappa(\gamma - r_t)}{[\rho r_t + \kappa\gamma]} - \frac{1}{2}\frac{(\rho\eta)^2}{[\rho r_t + \kappa\gamma]^2} \right] dt + \frac{\rho\eta}{[\rho r_t + \kappa\gamma]}dB_t \quad (23)$$

2.3.2.2 Claim on Capital

A claim on capital can be defined as an asset whose payoff is the future capital stock K_s . Using the stochastic discount factor we can find the price for a claim on the capital stock using the basic pricing equation:

$$P_{c,t} = E_t \left[\frac{\Lambda_s}{\Lambda_t} K_s \right]$$

Now using (17) and the SDF given by (7) together with the basic pricing equation (8) we obtain for the price, (as shown in the appendix)

$$P_{c,t} = K_t e^{-\rho(s-t)} \quad (24)$$

If we are interested in the price movement of such assets, or to be more precisely in the price movement of the asset class, the prices follow

$$dP_{c,t} = \frac{dK_t}{K_t} P_{c,t}$$

or

$$d\ln P_{c,t} = d\ln K_t = d\ln C_t$$

which states that the log price of the claim on capital behaves like the log capital stock. In other words, the instantaneous return on the claim can be interpreted as percentage changes in the capital stock, i.e., the controlled SDE driven by stochastic depreciation,

$$d\ln P_{c,t} = (r_t - \rho - \delta - \frac{1}{2}\sigma^2)dt + \sigma dZ_t$$

$$\begin{aligned}
d \ln P_{c,t} &= (r_t - \rho - \delta - \frac{1}{2}\sigma^2) dt + \sigma dZ_t \\
d \ln P_{d,t} &= \left[r_t - \rho - \delta - \frac{1}{2}\sigma^2 + \frac{\rho\kappa(\gamma - r_t)}{[\rho r_t + \kappa\gamma]} - \frac{1}{2} \frac{(\rho\eta)^2}{[\rho r_t + \kappa\gamma]^2} \right] dt \\
&\quad + \frac{\rho\eta}{[\rho r_t + \kappa\gamma]} dB_t + \sigma dZ_t \\
dr_t &= \kappa(\gamma - r_t)dt + \eta dB_t, \quad \text{where } r_t = r_t^f + \delta + \sigma^2
\end{aligned}$$

2.3.3 Combined Macro-Finance Framework

The formulation of the dividend claim gives some important insights into the behaviour of macro and financial variables as suggest by our model. Equation 18 states that the price of the dividend claim consists of a combination of macro and finance data as well as model parameters.

$$P_{d,t} = K_t \left[\frac{r_t - \gamma}{(\rho + \kappa)} + \frac{\gamma}{\rho} \right]$$

Rearranging terms

$$K_t = P_{d,t} \left[\frac{\rho^2 + \rho\kappa}{\rho r_t + \kappa\gamma} \right]$$

That is, in our macro-finance model, macroeconomic variables can be expressed completely in terms of financial data and parameters. Multiplying both sides of the above equation by ρ we can deduce an expression for consumption or by multiplying by r_t we obtain a financial expression for output. In a similar manner we can also derive an expression for the rental rate on physical capital, expressed in terms of macro-finance data and parameters.

$$r_t = \frac{P_{d,t}}{C_t}(\rho + \kappa) - \frac{\gamma\kappa}{\rho}$$

While this formulation yields an expression for the rental rate of physical capital in terms of both macro and financial variables, the availability of consumption data limits the frequency. Thus, in our model setting, given financial data and parameter values we can derive time series for macro economic variables at any desired frequency. Even though, this simple economic model is probably misspecified it still shows how macro-finance linkages and especially how macro and finance data can be evaluated in a joint and consistent framework.

Thus, the combined macro-finance system of estimation equations reads

$$d \ln(C_t) = \left((\rho + \kappa) \frac{P_{d,v}}{C_v} - \frac{\kappa\gamma}{\rho} - \rho - \delta - \frac{1}{2}\sigma^2 \right) dt + \sigma dZ_t \quad (25a)$$

$$\begin{aligned}
\ln(P_{d,t}/P_{d,t-\Delta}) &= (\rho + \kappa) \int_{t-\Delta}^t \frac{P_{d,v}}{C_v} dv - \left(\frac{\kappa\gamma}{\rho} + \kappa + \rho + \delta + \frac{1}{2}\sigma^2 \right) \Delta \\
&\quad + \frac{\gamma}{\rho} \int_{t-\Delta}^t \frac{C_v}{P_{d,v}} dv - \frac{1}{2} \frac{\eta^2}{(\rho + \kappa)^2} \int_{t-\Delta}^t \left(\frac{C_v}{P_{d,v}} \right)^2 dv + \varepsilon_{d,t}
\end{aligned} \quad (25b)$$

$$\frac{P_{d,t}}{C_t} = e^{-\kappa\Delta} \left(\frac{P_{d,t-\Delta}}{C_{t-\Delta}} \right) + (1 - e^{-\kappa\Delta}) \left(\frac{\gamma}{\rho} \right) + \varepsilon_{r,t} \quad (25c)$$

3 Estimation and Simulation Study

3.1 Specifying the Estimation Equations

In order to obtain robust results we use alternative estimation methods and also compare the resulting parameter estimates. As already pointed out in the introduction, the central procedures used in this paper are GMM and MEF estimation techniques. We apply optimal GMM as well as optimal MEF estimation and consider different numbers of conditional moment and parameter restrictions. A complete specification of the estimation equations can be found in the web appendix.

The equilibrium equations that we are using (in different combinations) to obtain the structural parameter estimates are

$$d \ln C_t = (r_t - \rho - \delta - \frac{1}{2}\sigma^2) dt + \sigma dZ_t \quad (26a)$$

$$d \ln P_{c,t} = (r_t - \rho - \delta - \frac{1}{2}\sigma^2) dt + \sigma dZ_t \quad (26b)$$

$$d \ln P_{d,t} = \left(r_t - \rho - \delta - \frac{1}{2}\sigma^2 + \frac{\rho\kappa(\gamma - r_t)}{(\rho r_t + \kappa\gamma)} - \frac{1}{2} \frac{(\rho\eta)^2}{(\rho r_t + \kappa\gamma)^2} \right) dt + \frac{\rho\eta}{(\rho r_t + \kappa\gamma)} dB_t + \sigma dZ_t \quad (26c)$$

$$d \ln Y_t = \left(\frac{\kappa\gamma}{r_t} - \frac{1}{2} \frac{\eta^2}{r_t^2} + r_t - \kappa - \rho - \delta - \frac{1}{2}\sigma^2 \right) dt \quad (26d)$$

$$+ \frac{\eta}{r_t} dB_t + \sigma dZ_t \quad (26e)$$

$$dr_t = \kappa(\gamma - r_t)dt + \eta dB_t, \quad \text{where } r_t = r_t^f + \delta + \sigma^2 \quad (26f)$$

3.2 Discrete-time Version of the Models

For the sake of clarity, the following sections will only show the derivations for the most central system of estimation equation. For a complete overview of all used estimators and estimation equations we refer the interest reader to our web appendix.

To account for the discrete-time character of the data we start from the systems of differential in section 2, integrate over $t \geq (t - \Delta)$, use exact solution whenever possible and arrive at exact discrete-time analogs based on observable variables.

The six structural parameters of the model are given by the vector

$$\phi = (\kappa, \gamma, \eta, \rho, \delta, \sigma)^T$$

We begin with the system using consumption, the dividend claim and the short rate. The complete derivations to the following results are shown in the appendix. The discrete version of this system reads.

$$\ln(C_t/C_{t-\Delta}) = \int_{t-\Delta}^t r_v^f dv - (\rho - \frac{1}{2}\sigma^2)\Delta + \varepsilon_{C,t} \quad (27a)$$

$$\ln(P_{d,t}/P_{d,t-\Delta}) = \int_{t-\Delta}^t r_v^f dv - (\rho - \frac{1}{2}\sigma^2)\Delta \quad (27b)$$

$$\begin{aligned} & + \rho\kappa \int_{t-\Delta}^t \left(\frac{\gamma - \delta - \sigma^2}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv \\ & - \rho\kappa \int_{t-\Delta}^t \left(\frac{r_v^f}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv \\ & - \frac{1}{2}(\rho\eta)^2 \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^2} dv + \varepsilon_{P_{d,t}} \\ r_t^f & = e^{-\kappa\Delta} r_{t-\Delta}^f + (1 - e^{-\kappa\Delta})(\gamma - \delta - \sigma^2) + \varepsilon_{r,t} \end{aligned} \quad (27c)$$

We define m_t the (3x1) vector of martingale difference sequence as

$$m_t = \begin{pmatrix} \ln(C_t/C_{t-\Delta}) - \int_{t-\Delta}^t r_v^f dv + (\rho - \frac{1}{2}\sigma^2)\Delta \\ \ln(P_{d,t}/P_{d,t-\Delta}) - \int_{t-\Delta}^t r_v^f dv + (\rho - \frac{1}{2}\sigma^2)\Delta - \rho\kappa \int_{t-\Delta}^t \left(\frac{\gamma - \delta - \sigma^2}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv \\ + \rho\kappa \int_{t-\Delta}^t \left(\frac{r_v^f}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv + \frac{1}{2}(\rho\eta)^2 \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^2} dv \\ r_t^f - e^{-\kappa\Delta} r_{t-\Delta}^f - (1 - e^{-\kappa\Delta})(\gamma - \delta - \sigma^2) \end{pmatrix} \quad (28)$$

Where the (3x1) vector of martingale increments, ε_t , is given by

$$\varepsilon_t = \begin{pmatrix} \varepsilon_{C,t} \\ \varepsilon_{d,t} \\ \varepsilon_{r,t} \end{pmatrix} = \begin{pmatrix} \sigma(Z_t - Z_{t-\Delta}) \\ \rho\eta \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]} dB_v + \sigma(Z_t - Z_{t-\Delta}) \\ \eta e^{-\kappa\Delta} \int_{t-\Delta}^t e^{\kappa(v-(t-\Delta))} dB_v \end{pmatrix} \quad (29)$$

As shown in the appendix, Ψ_t , the (3x3) conditional covariance matrix reads

$$\Psi_t = \begin{pmatrix} \sigma^2\Delta & \sigma^2\Delta & 0 \\ \sigma^2\Delta & (\rho\eta)^2\Delta/[\rho(r_{t-\Delta}^f + \delta + \sigma^2) + \kappa\gamma]^2 + \sigma^2\Delta & \rho\eta^2 e^{-\kappa\Delta}\Delta/[\rho(r_{t-\Delta}^f + \delta + \sigma^2) + \kappa\gamma] \\ 0 & \rho\eta^2 e^{-\kappa\Delta}\Delta/[\rho(r_{t-\Delta}^f + \delta + \sigma^2) + \kappa\gamma] & \eta^2(1 - e^{-2\kappa\Delta})/2\kappa \end{pmatrix} \quad (30)$$

Finally, the conditional mean of the parameter derivatives, using a first order deterministic Taylor expansion reads

$$\psi_t^T = \begin{pmatrix} 0 & \psi_{12} & \Delta e^{-\kappa\Delta}(r_{t-\Delta} - \gamma) \\ 0 & \psi_{22} & -(1 - e^{-\kappa\Delta}) \\ 0 & \psi_{32} & 0 \\ \Delta & \psi_{42} & 0 \\ 0 & \psi_{52} & (1 - e^{-\kappa\Delta}) \\ -\sigma\Delta & \psi_{62} & 2\sigma(1 - e^{-\kappa\Delta}) \end{pmatrix}$$

where the conditional mean of the parameter derivatives for the price of the claim on future dividends in the middle column is given by

$$\begin{aligned}
\psi_{12} &= -\rho(\gamma - \delta - \sigma^2) \mathbb{C}_1 + \rho\kappa\gamma(\gamma - \delta - \sigma^2) \mathbb{C}_2 - \gamma(\rho\eta)^2 \mathbb{C}_3 \\
&\quad + \rho \mathbb{C}_4 - \rho\kappa\gamma \mathbb{C}_5 \\
\psi_{22} &= -\rho\kappa \mathbb{C}_1 + \rho\kappa^2(\gamma - \delta - \sigma^2) \mathbb{C}_2 - \kappa(\rho\eta)^2 \mathbb{C}_3 \\
&\quad - \rho\kappa^2 \mathbb{C}_5 \\
\psi_{32} &= \eta\rho^2 \mathbb{C}_2 \\
\psi_{42} &= \Delta - \kappa(\gamma - \delta - \sigma^2) \mathbb{C}_1 + \rho\eta^2 \mathbb{C}_2 + \kappa \mathbb{C}_4 \\
&\quad + \rho\kappa(\gamma - \delta - \sigma^2) \mathbb{C}_6 - (\rho\eta)^2 \mathbb{C}_7 - \rho\kappa \mathbb{C}_8 \\
\psi_{52} &= \rho\kappa \mathbb{C}_1 + \rho^2\kappa(\gamma - \delta - \sigma^2) \mathbb{C}_2 - \rho^3\eta^2 \mathbb{C}_3 \\
&\quad - \rho^2\kappa \mathbb{C}_5 \\
\psi_{62} &= -\sigma\Delta + 2\rho\kappa\sigma \mathbb{C}_1 + 2\rho^2\kappa\sigma(\gamma - \delta - \sigma^2) \mathbb{C}_2 - 2\rho^3\eta^2\sigma \mathbb{C}_3 \\
&\quad - 2\rho^2\kappa\sigma \mathbb{C}_5
\end{aligned}$$

with the terms \mathbb{C}_i above defined as in appendix A.4.

As shown in the appendix, when replacing consumption with output the discrete version of the system reads

$$\ln(Y_t/Y_{t-\Delta}) = \int_{t-\Delta}^t r_v^f dv + \kappa\gamma \int_{t-\Delta}^t 1/(r_v^f + \delta + \sigma^2) dv \quad (31a)$$

$$-\frac{1}{2}\eta^2 \int_{t-\Delta}^t 1/(r_v^f + \delta + \sigma^2)^2 dv \quad (31b)$$

$$-(\kappa + \rho - \frac{1}{2}\sigma^2)\Delta + \varepsilon_{Y,t}$$

$$\begin{aligned}
\ln(P_{d,t}/P_{d,t-\Delta}) &= \int_{t-\Delta}^t r_v^f dv - (\rho - \frac{1}{2}\sigma^2)\Delta \\
&\quad + \rho\kappa \int_{t-\Delta}^t \left(\frac{\gamma - \delta - \sigma^2}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv \\
&\quad - \rho\kappa \int_{t-\Delta}^t \left(\frac{r_v^f}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv \\
&\quad - \frac{1}{2}(\rho\eta)^2 \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^2} dv + \varepsilon_{P_d,t}
\end{aligned}$$

$$r_t^f = e^{-\kappa\Delta} r_{t-\Delta}^f + (1 - e^{-\kappa\Delta})(\gamma - \delta - \sigma^2) + \varepsilon_{r,t} \quad (31c)$$

We define m_t the (3x1) vector of martingale difference sequence as

$$m_t = \begin{pmatrix} \ln(Y_t/Y_{t-\Delta}) - \int_{t-\Delta}^t r_v^f dv - \kappa\gamma \int_{t-\Delta}^t 1/(r_v^f + \delta + \sigma^2) dv \\ + \frac{1}{2}\eta^2 \int_{t-\Delta}^t 1/(r_v^f + \delta + \sigma^2)^2 dv + (\kappa + \rho - \frac{1}{2}\sigma^2)\Delta \\ \ln(P_{d,t}/P_{d,t-\Delta}) - \int_{t-\Delta}^t r_v^f dv + (\rho - \frac{1}{2}\sigma^2)\Delta - \rho\kappa \int_{t-\Delta}^t \left(\frac{\gamma - \delta - \sigma^2}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv \\ + \rho\kappa \int_{t-\Delta}^t \left(\frac{r_v^f}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv + \frac{1}{2}(\rho\eta)^2 \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^2} dv \\ r_t^f - e^{-\kappa\Delta}r_{t-\Delta}^f - (1 - e^{-\kappa\Delta})(\gamma - \delta - \sigma^2) \end{pmatrix} \quad (32)$$

with the (3x3) conditional covariance matrix

$$\Psi_t = \begin{pmatrix} \eta^2\Delta/(r_{t-\Delta}^f + \delta + \sigma^2)^2 + \sigma^2\Delta & \Psi_{t,12} & \Psi_{t,13} \\ \Psi_{t,21} & \Psi_{t,22} & \Psi_{t,23} \\ \Psi_{t,31} & \Psi_{t,32} & \eta^2(1 - e^{-2\kappa\Delta})/2\kappa \end{pmatrix} \quad (33)$$

where

$$\begin{aligned} \Psi_{t,12} &= \Psi_{t,21} = (\rho\eta^2\Delta)/[\rho(r_{t-\Delta}^f + \delta + \sigma^2)^2 + \kappa\gamma(r_{t-\Delta}^f + \delta + \sigma^2)] + \sigma^2\Delta \\ \Psi_{t,13} &= \Psi_{t,31} = \eta^2 e^{-\kappa\Delta}\Delta/(r_{t-\Delta}^f + \delta + \sigma^2) \\ \Psi_{t,22} &= (\rho\eta)^2\Delta/[\rho(r_{t-\Delta}^f + \delta + \sigma^2) + \kappa\gamma]^2 + \sigma^2\Delta \\ \Psi_{t,23} &= \Psi_{t,32} = \rho\eta^2 e^{-\kappa\Delta}\Delta/[\rho(r_{t-\Delta}^f + \delta + \sigma^2) + \kappa\gamma] \end{aligned}$$

3.3 Estimation Method

Applying GMM estimation to our model is relatively straightforward. The vector of Instruments, z_t , used in the GMM estimation consists of lagged right-hand variables. We consider both GMM using only first moments as well as second moments and additionally check our results by considering different parameter restrictions.

3.4 Simulation Study

In order to examine the small sample properties of our estimation procedures we conduct simulation studies for the different versions of the model. We simulate 25 years of data for the short rate, consumption, output and the dividend claim from the model. The median estimates as well as the interquartile range for 1000 replications are reported in the tables below. Additionally, the parameter values used in the data generating process are given in the first columns. These values are also highlighted by red bars in the histograms shown in this section. Above each table a short description of the used system settings can be found. At the beginning of each column the used building blocks of the model are specified. Here, the terms *Int*, *Claim*, *ClaimCap*, *Cons* and *Out* denote the used estimation equations for the interest rate, the dividend claim, the claim on capital, consumption and output respectively. We start our simulation study from a purely financial perspective and then add or remove additional macro variables. By doing so we are able to show how the parameter estimates behave when incorporating macro economic dynamics into our financial systems of estimation equations.

The simulation study suggest that all parameters except for the depreciation rate, δ , can be identified from financial data alone. Considering a macro-finance framework, however, allows for the identification of all 6 structural parameters. For each system of estimation equations table (1) shows which structural parameters can be identified when using the MEF estimation approach. Here, identified parameters are marked with a cross. With respect to the identification of parameters, the simulation study also highlights the benefits of using the MEF instead of the GMM approach. In line with the findings of Christensen, Posch & van der Wel (2016), the GMM approach is, at least in the practical implementation, unable to identify all 6 structural parameters.

In all considered systems of estimation equations the GMM estimation approach is unable to estimate all 6 structural parameters. Thus, we fix the stochastic depreciation rate, δ , at its true parameter value throughout the GMM simulation study and estimate (if identifiable) the remaining parameters. Furthermore, since the law of motion of the claim on capital, in our model setting, concises with the one for consumption, we do not list the corresponding systems separately in table (2).

Parameter Identification using Macro and Finance Data									
	Only Finance Model				Combined Macro-Finance Model				
	Sytem1 Int	System2 Int Claim	System3 Int ClaimCap	System4 Int Claim ClaimCap	System5 Int Claim Cons	System6 Int Claim Out	System7 Int Cons	System8 Int Cons Out	System9 Int Cons Out Claim
κ	X	X	X	X	X	X	X	X	X
γ	X	X	X	X	X	X	X	X	X
η	X	X	X	X	X	X	X	X	X
ρ		X	X	X	X	X	X	X	X
δ						X		X	X
σ		X	X	X	X	X	X	X	X

Table 1: Parameter Identification

Simulation Study: GMM Estimation							
(Median Estimates, Interquartile Range given below estimates)							
	Only Finance			Macro-Finance			
	Sytem1 Int	System2 Int Claim	Systems4/5 Int Claim ClaimCap/Cons	System6* Int Claim Out	System7/3 Int Cons/ClaimCap	System8 Int Cons Out	System9* Int Cons Out Claim
$\kappa = 0.2$	0.3563 (0.2859)	0.3766 (0.2700)	0.2121 (0.0641)	<i>0.3715</i> (<i>0.2809</i>)	0.3554 (0.2927)	0.3117 (0.2790)	<i>0.2934</i> (<i>0.1935</i>)
$\gamma = 0.1$	0.0996 (0.0127)	0.0999 (0.0124)	0.1000 (0.0129)	<i>0.0994</i> (<i>0.0126</i>)	0.1000 (0.0125)	0.0997 (0.0144)	<i>0.0999</i> (<i>0.0133</i>)
$\eta = 0.01$	0.0100 (0.0006)	0.0099 (0.0006)	0.0099 (0.0006)	<i>0.0100</i> (<i>0.0006</i>)	0.0100 (0.0005)	0.0097 (0.0006)	<i>0.0100</i> (<i>0.0006</i>)
$\rho = 0.03$		0.0297 (0.0058)	0.0297 (0.0059)	<i>0.0295</i> (<i>0.0058</i>)	0.0300 (0.0055)	0.0301 (0.0056)	<i>0.0299</i> (<i>0.0057</i>)
$\delta = 0.05$	0.05	0.05	0.05	0.05	0.05	0.05	0.05
$\sigma = 0.02$	0.02	0.0222 (0.0022)	0.0198 (0.0013)	<i>0.0225</i> (<i>0.0023</i>)	0.0197 (0.0011)	0.0194 (0.0012)	<i>0.0211</i> (<i>0.0015</i>)

Table 2: Simulation Study Results GMM Estimation

When performing GMM estimation of the small-scale finance version, System1, the simulation study suggest that we can identify 3 of the 5 model parameters contained in the estimation equation for the interest rate. When fixing δ and σ at their true values we obtain the results reported in the second column of the table above. While the estimates for γ and η are extremely close to their values used in the data generating process, the parameter capturing the speed of mean-reversion, κ is heavily upward biased. Nevertheless, this bias, as extensively discussed by Christensen, Posch & van der Wel (2016), is a common feature in the estimation of mean-reverting models. When adding the claim on future dividends to System1, we obtain System2. Now all 6 parameters are contained in the system. While the estimates for κ , γ and η remain nearly unaltered compared to System1, we are now able to obtain estimates for the time preference rate, ρ , as well as for the volatility of the stochastic depreciation rate, σ . While the former median estimate lies at its true value, the latter tends to be slightly upward biased. This is an important finding, as it highlights, that, given suitable real world data, nearly all structural parameters of the model can be estimated from financial data alone.

We now turn the effects of adding macro variables to System2. When adding the differential on consumption, the GMM approach is still unable to identify all 6 structural parameters. However, the upward bias in both, κ and σ vanishes, while at the same time time leaving the remaining 3 parameter estimates nearly unaltered at their true values. Hence, the GMM simulation study suggest, that considering a macro-finance framework improves the accuracy of the parameter estimates. An important finding for the empirical implementation are numerical problems encountered in the second stage of the optimal GMM estimation. When considering systems containing both the dividend claim and output, the second stage estimates heavily diverge from their first stage estimates in many instances. Nevertheless, this behaviour only occurs in case of GMM estimation. The reason for this, is the highly involved computation of the inverse covariance used in the second stage of the GMM estimation. Thus, columns 5 and 8 only show the first stage estimates.

The estimation study results for the MEF estimation approach in table (3) are in line with those obtained for GMM. Most notably, however, is the ability of the macro-finance models to identify all 6 structural parameters. Furthermore, there are no longer computational difficulties when it comes to estimate systems containing both output and claim. To shed some lights into the simulation study results, figure (1) plots the histograms for System2, Systems4/5 and System6. For the complete finance version, System1, and for the macro-finance version, Systems4/5, the histograms capture our previous findings. In the histograms (1a) for System2, the estimates of parameters κ and σ tend to be upward biased and we can identify 5 parameters. The histograms for Systems4/5 in (1b) highlight the reduction in the bias of the two parameters when incorporating consumption. Finally, for System6, figure (1c) shows the case where output is added to the finance formulation in System2. While the model is now able to

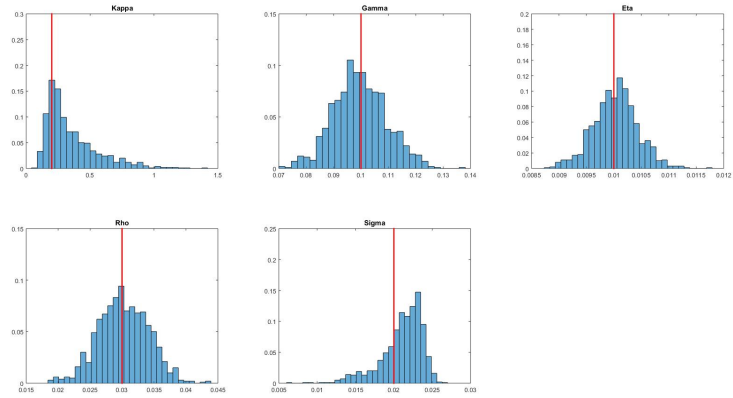
Simulation Study: MEF Estimation							
(Median Estimates, Interquartile Range given below estimates)							
Only Finance				Macro-Finance			
	Sytem1	System2	Systems4/5	System6	System7/3	System8	System9
	Int	Int Claim	Int Claim ClaimCap/Cons	Int Claim Out	Int Cons	Int Cons Out	Int Cons Out Claim
$\kappa = 0.2$	0.3507 (0.2725)	0.2844 (0.2401)	0.2018 (0.0462)	0.2800 (0.2104)	0.35354 (0.2685)	0.2983 (0.3768)	0.3506 (0.2722)
$\gamma = 0.1$	0.0995 (0.0127)	0.0991 (0.0129)	0.0994 (0.0134)	0.0993 (0.0132)	0.0992 (0.0127)	0.0101 (0.0148)	0.0993 (0.0128)
$\eta = 0.01$	0.0101 (0.0005)	0.0100 (0.0006)	0.0100 (0.0005)	0.0101 (0.0006)	0.0101 (0.0006)	0.0101 (0.0006)	0.010 (0.0005)
$\rho = 0.03$		0.0301 (0.0056)	0.0301 (0.0054)	0.0303 (0.0054)	0.0299 (0.0054)	0.0300 (0.0055)	0.0300 (0.0055)
$\delta = 0.05$	0.05	0.05	0.05	0.0500 (0.0015)	0.05	0.0500 (0.0019)	0.0500 (0.0016)
$\sigma = 0.02$	0.02	0.0216 (0.0031)	0.0200 (0.0011)	0.0215 (0.0028)	0.0200 (0.0011)	0.0200 (0.0011)	0.0200 (0.0011)

Table 3: Simulation Study Results MEF Estimation

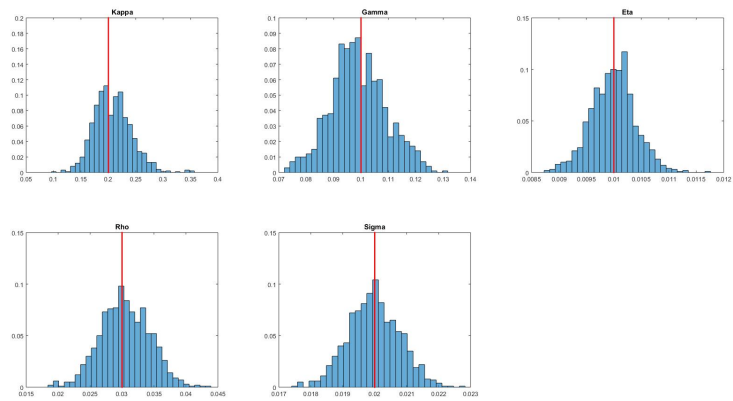
identify all 6 parameters, the previous upward bias in the estimates of κ and σ remains.

Figure 1: Histograms Simulation Study MEF Estimation

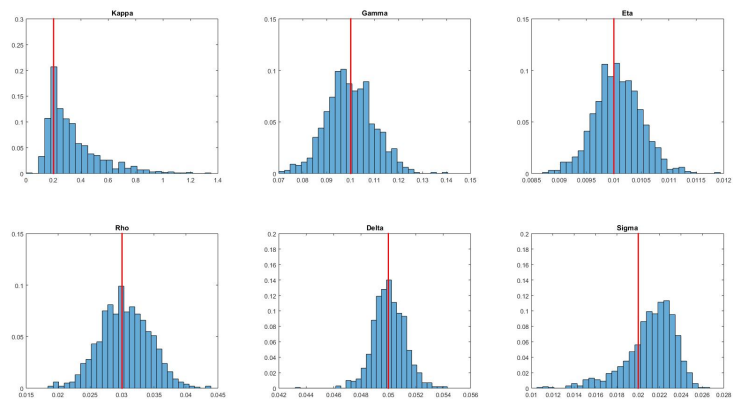
(a) Simulation Study, MEF-Estimation, System 2



(b) Simulation Study, MEF-Estimation, Systems 4/5



(c) Simulation Study, MEF-Estimation, System 6



4 Empirical Results and Data

4.1 Taking the Model to the Data

Our objective is the estimation of the structural parameters of our dynamic macroeconomic model using higher-frequency financial data. In this section we describe how to take our model to the data and offer economic intuition for using the price of the claim on future dividends. As shown in the web appendix we considered various alternative asset prices and returns. Those asset prices fully reflect the macro dynamics of our model but are unsuitable when it comes to consistently taking the model to the data. In this context the central problem is to find asset prices in the model that have a real world analogs. It is straight forward to incorporate output, consumption and interest rate data in our model estimation. To get the intuition for a real world analogue for the dividend claim, we will start with a simple example. Consider that an investor buys a broad defined stock index (a market portfolio) at period t . If he sells the stock index in the next period, his return is given by the sum of accumulated dividends up to this period, plus the price change of this index. In this context, the index across stocks represents the average production of firms in the economy and consists of stocks paying dividends. Like a stock index, the above defined claim on capital does not have an expiration date and gives the owner the right to all future dividends. Hence, to take our model to the data we turn to one of the most important indices worldwide, the S&P500 index and use this rich financial data from the stock market in our parameter estimation. As shown in the web appendix, we have to use the price rather than the return of the claim on future dividends to match it with stock return data.

In order to estimate the different systems of equilibrium equations of our model we need data on consumption, the short-term interest rate and on the price of the claim on future dividends. We consider the time period from January 1982 to December 2012. Data on consumption and the short rate is obtained from the Federal Reserve Economic Dataset (FRED). The monthly level of real Personal Consumption Expenditures (PCE) is used as a proxy for consumption. Following Christensen, Posch & van der Wel (2016), we use the 3-month interest rate, derived from US treasury bonds as proxy for the risk-free rate. For the claim on future dividends we use monthly data on the S&P500 obtained from the Center for Research in Security Prices (CRSP). This rich data set offers time series for differently computed returns and index levels ranging from 1925 up to January 2016. For our purpose the value weighted return of the S&P500 including dividends offers a suitable real world analogue for the price equation of the dividend claim.

In the empirical estimation we follow the same approach as in the simulation study. We start by estimating our two complete finance versions and successively add and remove additional macro and finance variables. The tables below are structured in the same manner as the once in the simulation study and show the results for the GMM and the MEF estimation respectively. Additionally, asymptotic t-statistics are given below the estimates.

Empirical Results: GMM-Estimation					
(Data: Value Weighted Return S&P500)					
(Asymptotic t-Statistics Given Below Estimates)					
	Only Finance			Macro-Finance	
	Sytem1	System2	Systems4	System6	System7
	Int	Int Claim	Int Claim Cons	Int Cons	Int Cons Out
κ	0.0432 (0.0282)	0.2061 (0.8199)	0.2034 (0.1569)	0.1800 (0.1289)	0.0517 (0.2086)
γ	0.0882 (0.0494)	0.1227 (1.1483)	0.1233 (1.0945)	0.1226 (0.8292)	0.1048 (0.5054)
η	0.0080 (0.9003)	0.0090 (0.9872)	0.0097 (1.2649)	0.0092 (1.0765)	0.0067 (0.8697)
ρ		0.0001 (0.0595)	0.0076 (0.1955)	0.0076 (0.1807)	0.0083 (0.2349)
δ	0.1	0.1	0.1	0.1	0.1
σ	0.02	0.02	0.0140 (0.6786)	0.0153 (0.8982)	0.01387 (1.1144)

One important observation that is present in all of the considered systems and estimation methods are the relatively low values of the parameter of time preference, ρ . This finding becomes especially clear when using stock market data in the estimation. The parameter estimates for ρ seem to be unrealistically low, as it suggest a time preference rate well below one percent.

As shown in the simulation study, the GMM estimation approach is able to identify κ , γ and η when considering only the equilibrium equation for the interest rate. Hence, we have to restrict the remaining 2 parameters of the model. As before, the parameter ρ is not contained in this small-scale finance system. The standard errors are quite high in this setting, causing the asymptotic t-statistics to be low.

Although the simulation study suggested otherwise, the empirical GMM estimation of System2 exhibit problems with the identification of σ . Thus, we restrict this parameter in the empirical GMM approach and estimate the remaining 4 parameters. Again, the estimate of ρ is unrealistically low. Nevertheless, introducing the dividend claim to model yields plausible parameter estimates for κ , γ and η that are in line with the once obtained in the macro-finance systems. As can be seen in column 4, adding consumption data to System2, yields plausible estimates for all parameters. However, we still have to rely on one parameter restriction. In line with the findings of the simulation study, consumption increases the accuracy of the estimates, at least in terms of asymptotic t-statistics.

As already mentioned above, the MEF estimation also yields extremely low estimates for the time preference rate, ρ . Furthermore, nearly all asymptotic t-statistics tend to be too low as well. When comparing the MEF and GMM estimation results, and by considering that nearly all systems of estimation equation yield similar parameter estimates, this finding is somewhat odd. To improve inference, it is necessary to go beyond asymptotic statistics.

Empirical Results: MEF-Estimation							
(Data: Value Weighted Return S&P500)							
(Asymptotic t-Statistics Given Below Estimates)							
	Only Finance			Macro-Finance			
	Sytem1	System2	Systems4	System5	System6	System7	System8
	Int	Int Claim	Int Claim Cons	Int Claim Out	Int Cons	Int Cons Out	Int Cons Out Claim
κ	0.1755 (37.799)	0.0009 (0.0034)	0.01867 (0.0204)	0.0726 (1.3738)	0.1152 (1.5372)	0.0987 (1.023)	0.0215 (0.0649)
γ	0.1222 (92.094)	0.1207 (0.0158)	0.1384 (0.1378)	0.0915 (0.5392)	0.1137 (0.4269)	0.1166 (0.9961)	0.0945 (0.0313)
η	0.0120 (3.3037)	0.0103 (0.0003)	0.0154 (0.0043)	0.0107 (0.1812)	0.0104 (0.0001)	0.0054 (0.0274)	0.0121 (0.0275)
ρ		0.00004 (0.0013)	0.002 (0.0332)	0.0063 (0.2684)	0.0142 (0.1076)	0.0136 (0.1300)	0.0001 (0.0026)
δ	0.1	0.1	0.1	0.1312 (1.0047)	0.1	0.0851 (0.2723)	0.0536 (0.0304)
σ	0.02	0.0129 (0.0016)	0.0146 (0.0033)	0.01641 (0.0090)	0.0250 (0.0104)	0.0199 (0.0037)	0.0121 (0.0017)

5 Conclusion

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A Appendix A

A.1 Properties and Derivations for the Stochastic Discount Factor

Starting from equation (6) we apply Ito's formula to obtain the evolution of $\ln(\Lambda_t)$:

$$\begin{aligned} d\ln(\Lambda_t) &= \frac{1}{\Lambda_t}(d\Lambda_t) - \frac{1}{2} \frac{1}{\Lambda_t^2}(d\Lambda_t)^2 \\ &= -(r_t - \delta - \frac{1}{2}\sigma^2)dt - \sigma dZ_t \end{aligned}$$

Integrating yields:

$$\int_t^s d\ln(\Lambda_v)dv = - \int_t^s (r_v - \delta - \frac{1}{2}\sigma^2)dv - \sigma \int_t^s dZ_v$$

From which we obtain the stochastic discount factor as the process given by equation (7).

Now to compute the expected value of the SDF we start from equation (?). Since this is an Ornstein-Uhlenbeck process we can find the solution by using a standard technique in differential equations as shown below.

$$\begin{aligned} e^{\kappa t}(dr_t + \kappa r_t)dt &= e^{\kappa t}\kappa\gamma dt + e^{\kappa t}\eta dB_t \\ \int_t^s (dr_t e^{\kappa u}) &= \int_t^s (d\gamma e^{\kappa u}) + \eta \int_t^s e^{\kappa u} dB_u \\ e^{\kappa s}r_s - e^{\kappa t}r_t &= e^{\kappa s}\gamma - e^{\kappa t}\gamma + \eta \int_t^s e^{\kappa u} dB_u \\ r_s &= e^{-\kappa(s-t)}r_t + (1 - e^{-\kappa(s-t)})\gamma + \eta e^{-\kappa(s-t)} \int_t^s e^{\kappa(u-t)} dB_u \end{aligned}$$

Note that in order to obtain the expected value of the stochastic discount factor we employ log-normality and compute

$$\ln E_t [e^{\ln(\Lambda_s) - \ln(\Lambda_t)}] = E_t[\ln(\Lambda_s) - \ln(\Lambda_t)] + \frac{1}{2}Var_t[\ln(\Lambda_s) - \ln(\Lambda_t)] \quad (34)$$

We can now plug our solution for r_s into our log expression for the stochastic discount factor and obtain

$$\begin{aligned} \ln(\Lambda_s) - \ln(\Lambda_t) &= - \int_t^s r_v dv + \int_t^s (\delta + \frac{1}{2}\sigma^2)dv - \sigma \int_t^s dZ_v \\ &= - \int_t^s (e^{-\kappa(v-t)}r_t + (1 - e^{-\kappa(v-t)})\gamma - \delta - \frac{1}{2}\sigma^2)dv \\ &\quad - \eta \int_t^s e^{-\kappa(v-t)} \int_t^v e^{\kappa(u-t)} dB_u dv - \sigma \int_t^s dZ_v \end{aligned}$$

Reversing the order of integration and evaluating the ds integrals yield

$$\ln(\Lambda_s) - \ln(\Lambda_t) = -\frac{r_t - \gamma}{\kappa}(1 - e^{-\kappa(s-t)}) - (\gamma - \delta - \frac{1}{2}\sigma^2)(s-t) - \frac{\eta}{\kappa} \int_t^s (1 - e^{-\kappa(s-u)}) dB_u - \sigma \int_t^s dZ_v$$

Inspection of the last two integrals give rise to a normally distributed random variable with mean zero and variance

$$\begin{aligned} \text{Var}_t[\ln(\Lambda_s) - \ln(\Lambda_t)] &= \int_t^s \left(\frac{\eta}{\kappa}(1 - e^{-\kappa(s-u)}) \right)^2 du + \int_t^s \sigma^2 du \\ &= \left(\left(\frac{\eta}{\kappa} \right)^2 + \sigma^2 \right) (s-t) - 2\frac{\eta^2}{\kappa^3}(1 - e^{-\kappa(s-t)}) + \frac{\eta^2}{2\kappa^3}(1 - e^{-2\kappa(s-t)}) \end{aligned}$$

And

$$E_t[\ln(\Lambda_s) - \ln(\Lambda_t)] = -\frac{r_t - \gamma}{\kappa}(1 - e^{-\kappa(s-t)}) - (\gamma - \delta - \frac{1}{2}\sigma^2)(s-t)$$

Thus by plugging in we conclude

$$\ln E_t [e^{\ln \Lambda_s - \ln \Lambda_t}] = -\left(\frac{r_t - \gamma}{\kappa} + \frac{\eta^2}{\kappa^3} \right) (1 - e^{-\kappa(s-t)}) - \left(\gamma - \delta - \sigma^2 - \frac{1}{2} \frac{\eta^2}{\kappa^2} \right) (s-t) + \frac{\eta^2}{4\kappa^3} (1 - e^{-2\kappa(s-t)})$$

From which we obtain (??).

A.2 Properties and Derivations for the Claim on Future Dividends

To obtain the expression for period k 's capital stock given by (17) we use the SDF given by (7) together with the basic pricing equation (8). We start with the equation for the evolution of capital, where we substitute I_t to express dK_t in terms of C_t and r_t

$$\begin{aligned} dK_t &= (I_t - \delta K_t) + \sigma K_t dZ_t, \\ &= (r_t K_t - C_t - \delta K_t) dt + \sigma K_t dZ_t \end{aligned}$$

where $I_t = Y_t K_t - C_t = r_t K_t - C_t$.

Now we use Ito's formula to derive an expression for $d \ln(K_t)$

$$\begin{aligned} d \ln(K_t) &= \frac{1}{K_t} dK_t - \frac{1}{2} \frac{1}{K_t^2} (dK_t)^2 \\ &= \frac{1}{K_t} (r_t K_t - C_t - \delta K_t) + \frac{1}{K_t} \sigma K_t dZ_t - \frac{1}{2} \frac{1}{K_t^2} \sigma^2 K_t^2 dt \\ &= \left(r_t - \frac{C_t}{K_t} - \delta - \frac{1}{2} \sigma^2 \right) dt + \sigma dZ_t \\ &= \left(r_t - \rho - \delta - \frac{1}{2} \sigma^2 \right) dt + \sigma dZ_t \end{aligned}$$

Where we used the closed-form solution $\rho = C_t/K_t$.

Now, to obtain an expression for K_s integrate over t to s :

$$\begin{aligned}\int_t^s d \ln(K_t) dt &= \int_t^s (r_v - \rho - \delta - \frac{1}{2}\sigma^2) dv + \sigma \int_t^s dZ_v \\ \ln(K_s) - \ln(K_t) &= \int_t^s [r_v - \rho - \delta - \frac{1}{2}\sigma^2] dv + \sigma \int_t^s dZ_v \\ K_s &= K_t e^{\int_t^s (r_v - \rho - \delta - \frac{1}{2}\sigma^2) dv + \sigma \int_t^s dZ_v}\end{aligned}$$

which is the same as (17).

Now to obtain an expression for the capital stock in period s recall that in the AK-Vasicek model A_t is equal to r_t . Thus, the evolution of TFP is captured by the evolution of r_t . Furthermore, due to the AK-specification we have $Y_t = A_t K_t$. The evolution of Y_t , can be described in terms of dr_t and dK_t

$$\begin{aligned}dK_t &= (r_t K_t - C_t - \delta K_t) dt + \sigma K_t dZ_t \\ dr_t &= \kappa(\gamma - r_t) dt + \eta dB_t\end{aligned}$$

together with the Euler equation, or using the analytical solution $C_t = \rho K_t$.

Now, using Ito's formula to derive an expression for $d \ln(K_t r_t)$ we arrive at (18) by computing

$$\begin{aligned}d \ln(K_t r_t) &= \frac{1}{K_t} (dK_t) - \frac{1}{2} \frac{1}{K_t^2} (dK_t)^2 + \frac{1}{r_t} (dr_t) - \frac{1}{2} \frac{1}{r_t^2} (dr_t)^2 \\ d \ln(Y_t) &= (r_t - \delta - \rho - \frac{1}{2}\sigma^2 + \frac{\kappa\gamma}{r_t} - \kappa - \frac{1}{2} \frac{\eta^2}{r_t^2}) dt + \sigma dZ_t + \frac{\eta}{r_t} dB_t \\ Y_s &= Y_t e^{\kappa\gamma \int_t^s \frac{1}{r_v} dv - \frac{\eta^2}{2} \int_t^s \frac{1}{r_v^2} dv + \int_t^s (r_v - \delta - \rho - \kappa - \frac{1}{2}\sigma^2) dv + \sigma \int_t^s dZ_v + \int_t^s \frac{\eta}{r_v} dB_v}\end{aligned}$$

The last equation is analogous to (15) in Christensen, Posch & van der Wel (2016).

In the derivation of (19), however, we will exploit the Ornstein Uhlenbeck specification. Considering the Vasicek specification of the interest rate we know that:

$$A_s = r_s = e^{-\kappa(s-t)} \left(r_t + (e^{\kappa(s-t)} - 1)\gamma + \eta \int_t^s e^{\kappa(u-t)} dB_u \right)$$

While period's s capital stock is given by:

$$K_s = K_t e^{\int_t^s (r_v - \rho - \delta - \frac{1}{2}\sigma^2) dv + \sigma \int_t^s dZ_v}$$

Hence, Y_s is given by

$$A_s K_s = \left[K_t r_t + K_t \gamma e^{\kappa(s-t)} - K_t \gamma + K_t \eta \int_t^s e^{\kappa(u-t)} dB_u \right] e^{\int_t^s (r_v - \delta - \rho - \kappa - \frac{1}{2}\sigma^2) dv + \sigma \int_t^s dZ_v}$$

Starting from equation (2.3.3) note that

$$\begin{aligned}
P_{d,t} &= K_t \left[\frac{r_t - \gamma}{(\rho + \kappa)} + \frac{\gamma}{\rho} \right] \\
&= dK_t \left[\frac{r_t - \gamma}{(\rho + \kappa)} + \frac{\gamma}{\rho} \right] + \frac{K_t}{(\rho + \kappa)} dr_t \\
&= P_{d,t} \frac{dK_t}{K_t} + \frac{K_t}{(\rho + \kappa)} dr_t \\
&= P_{d,t} [r_t - \rho - \delta] dt + P_{d,t} \sigma dZ_t + \frac{K_t}{(\rho + \kappa)} [\kappa(\gamma - r_t)] dt + \frac{K_t}{(\rho + \kappa)} \eta dB_t
\end{aligned}$$

To obtain an expression for the last term in terms of $P_{d,t}$ note that:

$$\frac{K_t}{\rho + \kappa} = \frac{\rho P_{d,t}}{\rho r_t + \kappa \gamma}$$

or

$$\frac{K_t}{\rho + \kappa} = P_{d,t} \left[\frac{1}{r_t + \frac{\kappa \gamma}{\rho}} \right]$$

Plugging in yields:

$$dP_{d,t} = P_{d,t} [r_t - \rho - \delta] dt + P_{d,t} \sigma dZ_t + P_{d,t} \frac{[\rho \kappa (\gamma - r_t)]}{\rho r_t + \kappa \gamma} dt + P_{d,t} \frac{\rho \eta}{\rho r_t + \kappa \gamma} dB_t$$

or in logs:

$$d \ln P_{d,t} = d \ln K_t + \left[\frac{\rho \kappa (\gamma - r_t)}{[\rho r_t + \kappa \gamma]} - \frac{1}{2} \frac{(\rho \eta)^2}{[\rho r_t + \kappa \gamma]^2} \right] dt + \frac{\rho \eta}{[\rho r_t + \kappa \gamma]} dB_t$$

A.3 Derivations of the Discrete time formulations

In this section we will derive the discrete time formulations for the system of equilibrium equations given by (26). We start from the baseline model and substitute the equation for log output by the equation for the log price of the claim on future dividends

Hence we have for the claim on future dividends

$$\begin{aligned}
d \ln P_{d,t} &= d \ln K_t + \left[\frac{\rho \kappa (\gamma - r_t)}{[\rho r_t + \kappa \gamma]} - \frac{1}{2} \frac{(\rho \eta)^2}{[\rho r_t + \kappa \gamma]^2} \right] dt + \frac{\rho \eta}{[\rho r_t + \kappa \gamma]} dB_t \\
&= \left[\frac{\rho \kappa (\gamma - r_t^f - \delta - \sigma^2)}{[\rho (r_t^f + \delta + \sigma^2) + \kappa \gamma]} - \frac{1}{2} \frac{(\rho \eta)^2}{[\rho (r_t^f + \delta + \sigma^2) + \kappa \gamma]^2} \right] dt \\
&\quad + (r_t^f - \rho + \frac{1}{2} \sigma^2) dt + \frac{\rho \eta}{[\rho (r_t^f + \delta + \sigma^2) + \kappa \gamma]} dB_t + \sigma dZ_t
\end{aligned}$$

now integrating over $(t - \Delta)$ to t we obtain

$$\begin{aligned} \ln(P_{d,t}/P_{d,t-\Delta}) &= \int_{t-\Delta}^t r_v^f dv - (\rho - \frac{1}{2}\sigma^2)\Delta + \rho\kappa \int_{t-\Delta}^t \frac{\gamma - r_v^f - \delta - \sigma^2}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} dv \\ &\quad - \frac{1}{2}(\rho\eta)^2 \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^2} dv \\ &\quad + \rho\eta \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]} dB_v + \sigma \int_{t-\Delta}^t dZ_v \end{aligned}$$

Since we just want to substitute this equation in the baseline model we obtain the discrete version of (26) as

$$\begin{aligned} \ln(C_t/C_{t-\Delta}) &= \int_{t-\Delta}^t r_v^f dv - (\rho - \frac{1}{2}\sigma^2)\Delta + \varepsilon_{C,t} \\ \ln(P_{d,t}/P_{d,t-\Delta}) &= \int_{t-\Delta}^t r_v^f dv - (\rho - \frac{1}{2}\sigma^2)\Delta + \rho\kappa \int_{t-\Delta}^t \left(\frac{\gamma - r_v^f - \delta - \sigma^2}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv \\ &\quad - \frac{1}{2}(\rho\eta)^2 \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^2} dv + \varepsilon_{P_d,t} \\ r_t^f &= e^{-\kappa\Delta} r_{t-\Delta}^f + (1 - e^{-\kappa\Delta})(\gamma - \delta - \sigma^2) + \varepsilon_{r,t} \end{aligned}$$

where the martingale increments are defined by

$$\begin{aligned} \varepsilon_{C,t} &\equiv \sigma(Z_t - Z_{t-\Delta}) \\ \varepsilon_{d,t} &\equiv \rho\eta \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]} dB_v + \sigma \int_{t-\Delta}^t dZ_v \\ \varepsilon_{r,t} &\equiv \eta e^{-\kappa\Delta} \int_{t-\Delta}^t e^{\kappa(v-(t-\Delta))} dB_v \end{aligned}$$

Using the above calculations together with the expression for log output of the discrete

version of system (??) we obtain the discrete version of system (??) as

$$\begin{aligned}
\ln(Y_t/Y_{t-\Delta}) &= \int_{t-\Delta}^t r_v^f dv + \kappa\gamma \int_{t-\Delta}^t 1/(r_v^f + \delta + \sigma^2) dv \\
&\quad - \frac{1}{2}\eta^2 \int_{t-\Delta}^t 1/(r_v^f + \delta + \sigma^2)^2 dv \\
&\quad - (\kappa + \rho - \frac{1}{2}\sigma^2)\Delta + \varepsilon_{Y,t} \\
\ln(P_{d,t}/P_{d,t-\Delta}) &= \int_{t-\Delta}^t r_v^f dv - (\rho - \frac{1}{2}\sigma^2)\Delta \\
&\quad + \rho\kappa \int_{t-\Delta}^t \left(\frac{\gamma - \delta - \sigma^2}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv \\
&\quad - \rho\kappa \int_{t-\Delta}^t \left(\frac{r_v^f}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv \\
&\quad - \frac{1}{2}(\rho\eta)^2 \int_{t-\Delta}^t \frac{1}{\left[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma \right]^2} dv + \varepsilon_{P_{d,t}} \\
r_t^f &= e^{-\kappa\Delta} r_{t-\Delta}^f + (1 - e^{-\kappa\Delta})(\gamma - \delta - \sigma^2) + \varepsilon_{r,t}
\end{aligned}$$

and define the vector of martingale increments as

$$\varepsilon_t = \begin{pmatrix} \varepsilon_{Y,t} \\ \varepsilon_{d,t} \\ \varepsilon_{r,t} \end{pmatrix} = \begin{pmatrix} \int_{t-\Delta}^t \frac{\eta}{[r_v^f + \delta + \sigma^2]} dB_v + \sigma \int_{t-\Delta}^t dZ_v \\ \rho\eta \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]} dB_v + \sigma \int_{t-\Delta}^t dZ_v \\ \eta e^{-\kappa(s-t)} \int_{t-\Delta}^t e^{\kappa(v-(t-\Delta))} dB_v \end{pmatrix}$$

A.4 MEF

The matrix of parameter derivatives reads

$$\Phi_t = \begin{pmatrix} 0 & \Phi_{12} & \Delta e^{-\kappa\Delta}(r_{t-\Delta} - \gamma) \\ 0 & \Phi_{22} & -(1 - e^{-\kappa\Delta}) \\ 0 & \Phi_{32} & 0 \\ \Delta & \Phi_{42} & 0 \\ 0 & \Phi_{52} & (1 - e^{-\kappa\Delta}) \\ -\sigma\Delta & \Phi_{62} & 2\sigma(1 - e^{-\kappa\Delta}) \end{pmatrix}$$

where

$$\begin{aligned}
\Phi_{12} = \frac{\partial m_2}{\partial \kappa} &= -\rho(\gamma - \delta - \sigma^2) \int_{t-\Delta}^t \left(\frac{1}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv - \gamma(\rho\eta)^2 \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^3} dv \\
&\quad + \rho\kappa\gamma(\gamma - \delta - \sigma^2) \int_{t-\Delta}^t \left(\frac{1}{(\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma)^2} \right) dv \\
&\quad + \rho \int_{t-\Delta}^t \left(\frac{r_v^f}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv - \rho\kappa\gamma \int_{t-\Delta}^t \left(\frac{r_v^f}{(\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma)^2} \right) dv
\end{aligned}$$

$$\begin{aligned}\Phi_{22} = \frac{\partial m_2}{\partial \gamma} &= -\rho\kappa \int_{t-\Delta}^t \left(\frac{1}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv + \rho\kappa^2(\gamma - \delta - \sigma^2) \int_{t-\Delta}^t \left(\frac{1}{(\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma)^2} \right) dv \\ &\quad - \rho\kappa^2 \int_{t-\Delta}^t \left(\frac{r_v^f}{(\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma)^2} \right) dv - \kappa(\rho\eta)^2 \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^3} dv\end{aligned}$$

$$\Phi_{32} = \frac{\partial m_2}{\partial \eta} = \eta\rho^2 \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^2} dv$$

$$\begin{aligned}\Phi_{42} = \frac{\partial m_2}{\partial \rho} &= -\kappa(\gamma - \delta - \sigma^2) \int_{t-\Delta}^t \left(\frac{1}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv \\ &\quad + \rho\kappa(\gamma - \delta - \sigma^2) \int_{t-\Delta}^t \left(\frac{r_v^f + \delta + \sigma^2}{(\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma)^2} \right) dv \\ &\quad + \kappa \int_{t-\Delta}^t \frac{r_v^f}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]} dv - \rho\kappa \int_{t-\Delta}^t \frac{r_v^f(r_v^f + \delta + \sigma^2)}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^2} dv + \Delta \\ &\quad + \rho\eta^2 \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^2} dv - (\rho\eta)^2 \int_{t-\Delta}^t \frac{r_v^f + \delta + \sigma^2}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^3} dv\end{aligned}$$

$$\begin{aligned}\Phi_{52} = \frac{\partial m_2}{\partial \delta} &= \rho\kappa \int_{t-\Delta}^t \left(\frac{1}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv + \rho^2\kappa(\gamma - \delta - \sigma^2) \int_{t-\Delta}^t \left(\frac{1}{(\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma)^2} \right) dv \\ &\quad - \rho^2\kappa \int_{t-\Delta}^t \frac{r_v^f}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^2} dv - \rho^3\eta^2 \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^3} dv\end{aligned}$$

$$\begin{aligned}\Phi_{62} = \frac{\partial m_2}{\partial \sigma} &= 2\rho\kappa\sigma \int_{t-\Delta}^t \left(\frac{1}{\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma} \right) dv + 2\rho^2\kappa\sigma(\gamma - \delta - \sigma^2) \int_{t-\Delta}^t \left(\frac{1}{(\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma)^2} \right) dv \\ &\quad - 2\rho^2\kappa\sigma \int_{t-\Delta}^t \frac{r_v^f}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^2} dv - 2\rho^3\eta^2\sigma \int_{t-\Delta}^t \frac{1}{[\rho(r_v^f + \delta + \sigma^2) + \kappa\gamma]^3} dv - \sigma\Delta\end{aligned}$$

$$\mathbb{C}_1 = \left(\frac{\Delta}{\rho r_{t-\Delta} + \kappa\gamma} - \left(\frac{\kappa(\gamma - r_{t-\Delta})\rho}{(\rho r_{t-\Delta} + \kappa\gamma)^2} - \frac{\eta^2 \rho^2}{(\rho r_{t-\Delta} + \kappa\gamma)^3} \right) \frac{1}{2} \Delta^2 \right)$$

$$\mathbb{C}_2 = \left(\frac{\Delta}{(\rho r_{t-\Delta} + \kappa\gamma)^2} - \left(\frac{\kappa(\gamma - r_{t-\Delta})2\rho}{(\rho r_{t-\Delta} + \kappa\gamma)^3} - \frac{\eta^2 3\rho^2}{(\rho r_{t-\Delta} + \kappa\gamma)^4} \right) \frac{1}{2} \Delta^2 \right)$$

$$\mathbb{C}_3 = \left(\frac{\Delta}{(\rho r_{t-\Delta} + \kappa\gamma)^3} - \left(\frac{\kappa(\gamma - r_{t-\Delta})3\rho}{(\rho r_{t-\Delta} + \kappa\gamma)^4} - \frac{\eta^2 6\rho^2}{(\rho r_{t-\Delta} + \kappa\gamma)^5} \right) \frac{1}{2} \Delta^2 \right)$$

$$\mathbb{C}_4 = \left(\frac{(r_{t-\Delta} - \delta - \sigma^2)\Delta}{(\rho r_{t-\Delta} + \kappa\gamma)} + \left(\frac{\kappa(\gamma - r_{t-\Delta})(\kappa\gamma + \rho\delta + \rho\sigma^2)}{(\rho r_{t-\Delta} + \kappa\gamma)^2} - \frac{\eta^2(\rho\kappa\gamma + \rho^2\delta + \rho^2\sigma^2)}{(\rho r_{t-\Delta} + \kappa\gamma)^3} \right) \frac{1}{2} \Delta^2 \right)$$

$$\mathbb{C}_5 = \left(\frac{(r_{t-\Delta} - \delta - \sigma^2)\Delta}{(\rho r_{t-\Delta} + \kappa\gamma)^2} + \left(\frac{\kappa(\gamma - r_{t-\Delta})(\kappa\gamma - \rho r_{t-\Delta} + 2\rho\delta + 2\rho\sigma^2)}{(\rho r_{t-\Delta} + \kappa\gamma)^3} + \frac{\eta^2(\rho^2 r_{t-\Delta} - 2\rho\kappa\gamma - 3\rho^2\delta - 3\rho^2\sigma^2)}{(\rho r_{t-\Delta} + \kappa\gamma)^4} \right) \right)$$

$$\mathbb{C}_6 = \left(\frac{r_{t-\Delta}\Delta}{(\rho r_{t-\Delta} + \kappa\gamma)^2} + \left(\frac{\kappa(\gamma - r_{t-\Delta})(\kappa\gamma - \rho r_{t-\Delta})}{(\rho r_{t-\Delta} + \kappa\gamma)^3} + \frac{\eta^2(\rho^2 r_{t-\Delta} - 2\rho\kappa\gamma)}{(\rho r_{t-\Delta} + \kappa\gamma)^4} \right) \frac{1}{2}\Delta^2 \right)$$

$$\mathbb{C}_7 = \left(\frac{r_{t-\Delta}\Delta}{(\rho r_{t-\Delta} + \kappa\gamma)^3} + \left(\frac{\kappa(\gamma - r_{t-\Delta})(\kappa\gamma - 2\rho r_{t-\Delta})}{(\rho r_{t-\Delta} + \kappa\gamma)^4} + \frac{3\eta^2(\rho^2 r_{t-\Delta} - \rho\kappa\gamma)}{(\rho r_{t-\Delta} + \kappa\gamma)^5} \right) \frac{1}{2}\Delta^2 \right)$$

$$\begin{aligned} \mathbb{C}_8 &= \frac{(r_{t-\Delta}^2 - r_{t-\Delta}\delta - r_{t-\Delta}\sigma^2)\Delta}{(\rho r_{t-\Delta} + \kappa\gamma)^2} \\ &+ \left(\frac{\kappa(\gamma - r_{t-\Delta})(r_{t-\Delta}(\rho\delta + \sigma^2\rho + 2\kappa\gamma) - \delta\kappa\gamma - \sigma^2\kappa\gamma)}{(\rho r_{t-\Delta} + \kappa\gamma)^3} + \frac{\eta^2(\kappa\gamma(\kappa\gamma + 2\rho\delta + 2\sigma^2\rho) - \rho r_{t-\Delta}(\rho\delta + \sigma^2\rho))}{(\rho r_{t-\Delta} + \kappa\gamma)^4} \right) \end{aligned}$$