Vector Quantile Autoregression: A Random Coefficient Approach

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Abstract

Vector autoregressions (VARs) are important statistical tools for empirical analysis. I develop a statistical framework where multiple economic shocks can affect the location, scale and shape of the entire conditional distribution of the multiple time-series of the responses, while in the constant coefficient VAR models shocks only affect the location. To achieve this goal, the multiple dynamic time-series data-generating process with the parameters being affine functions of random variables is introduced. Furthermore, I introduce the novel vector quantile autoregression (VQAR) that relates the vector of autoregressive quantile processes to its lagged values and propose a procedure for identifying the structural quantile shocks, the type of shock that occurs with a certain probability. I introduce the quantile impulse response functions (QIRFs) as a main device for estimating the impact and transmission of the structural quantile shocks. Asymptotic properties are discussed and bootstrap procedures are introduced for the inference purposes.

Keywords: Random parameter model, conditional distribution function, quantile regression, bootstrap confidence interval, companion form.

JEL: C22,C32.

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1 Introduction

Vector autoregressions (VARs) are important tools for policy analysis in economics (Lütkepohl, 2005). While most economic time series displays systematic asymmetric dynamics over time, appropriate VAR models are needed to explain and model such phenomena.[1] Using formal terminology, I treat asymmetries as an outcome of how a distribution of the vector of outcome variables $\boldsymbol{y}$ is affected by its lagged values and structural shocks. This paper shows that the goal can be achieved by modeling multiple dynamic conditional quantile functions, whose autoregressive slope parameters vary with quantiles $\tau \in (0, 1)$. The main goal of this paper is to provide statistical framework for estimation and inference procedures to study the entire distribution of $\boldsymbol{y}$ and its functionals in a dynamic setting using regression quantiles by Koenker and Bassett (1978).

This paper contributes to the literature on dynamic time-series models in several ways. First, it provides statistical framework for modeling the location, scale and shape of the conditional density of $\boldsymbol{y}$. Building on Koenker and Xiao (2006), I develop a dynamic model with an affine random coefficient specification as a tool for generating the τ-dependent autoregressive coefficients as a result of monotonic transformation.[2] This specification represents a useful modification to Koenker and Xiao (2006) since it relates shocks of the data-generating process to the vector of dynamic conditional quantile functions. There is a long list of theoretical contributions to the linear quantile autoregression (see Koenker and Xiao, 2005, and the references therein). However, Koenker and Xiao (2006) rigorously treat the problem of having a valid data-generating process behind the quantile autoregressions with the τ-dependent slope effects.

Second, starting from quantile and least squares parameter estimates, using Cholesky scheme I propose an algorithm to obtain structural shocks of the outcome variable. Under standard empirical identifying conditions, this procedure explores the linear interrelation between the parameters of the data-generating and conditional quantile processes. This procedure is general enough to accommodate many popular forms of identification.

[1] For classical references see Neftci (1984), Beaudry and Koop (1993), Enders and Granger (1998), among others.

[2] This approach to random coefficient modeling is much in the spirit of Swamy (1970).

Third, I introduce the vector quantile autoregression (VQAR), a companion form for the vector of dynamic quantile functions with the parameters being products of the time-varying and quantile specific coefficients. The VQAR is of independent economic interest since it relates the lagged value of quantile functions to its future value. Therefore I generalize classical VAR models and now the impact of shocks to the entire conditional distribution can be traced dynamically over the certain horizon. Formally this goal is achieved by deriving the quantile impulse response functions (QIRFs), recently introduced by Chavleishvili and Manganelli (2017), and using identified structural shocks to construct it.[3]

This paper contains two sets of theoretical results. First, under standard regularity conditions I show that the limiting properties of the parameters and QIRFs are to be established analytically. In particular, before the parameter estimates are discussed and then asymptotic properties of their functionals can be established using the Delta method. Second, for ease of inference, I propose asymptotically valid bootstrap procedure for inference on the QIRFs and the parameters of interest.

Following notation is used throughout the paper. For some matrix $\boldsymbol{\Phi}$, the scalar $(\boldsymbol{\Phi})_{ij}$ corresponds to its ij-th element and a row vector $(\boldsymbol{\Phi})_{i.}$ gives its i-th row. Vectors $\mathbf{0}_K$ and $\boldsymbol{\imath}_K$ are $K \times 1$ vectors of zeros and ones. I write $\sim$ to denote convergence in distribution.

The rest of the paper is organized as follows. Section 2 presents theoretical setting for the general form of the VARs with time-varying parameters, introduces related conditional quantile functions and derives the VQAR representation for them. In Section 3 I explicitly model parameters of the data-generating process and design a strategy for uncovering the structural shocks. Section 4 contains the main analytical results in terms of the QIRFs. In Section 5 I discuss limiting properties of QIRFs and propose bootstrap algorithm for the ease of inference. Section 6 designs simulations to study validity of the framework. Section 6 presents empirical results. Some of the technical results are relegated to Appendix.

[3]Chavleishvili and Manganelli (2017) considers the QIRFs in the context of the multivariate financial returns processes with the generalized autoregressive conditional heteroscedasticity (GARCH) dynamics as its conditional variance process.

2 The framework

I study a conditional distribution of the multiple time-series of observations. The goal is achieved by introducing the dynamic conditional vector quantile functions with the quantile-specific parameters. The vector quantile autoregression (VQAR) is introduced to study the impact and propagation of shocks.

Suppose, for a finite lag order $p \geq 1$, the objective is to explore the distribution of a finitely dimensioned $K \times 1$ vector of real-valued random variables $\boldsymbol{y}_t$ conditional on its past realizations $\boldsymbol{z}_{t-1} \equiv (\boldsymbol{y}'_{t-1}, \boldsymbol{y}'_{t-2}, \ldots, \boldsymbol{y}'_{t-p})'$ and subject to the serially uncorrelated $K \times 1$ vector of innovations $\boldsymbol{\varepsilon}_t | \boldsymbol{z}_{t-1} \sim i.i.d.(\boldsymbol{0}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ is a $K \times K$ positive definite matrix. The goal is to develop a tool for structural analysis of the impact of $\boldsymbol{z}_{t-1}$ and the realizations in $\boldsymbol{\varepsilon}_t$ on the entire conditional distribution of $\boldsymbol{y}_t$ and therefore to analyze the values of $\boldsymbol{y}_t$ that occur with different regularities.[4]

The central concept of this paper is a quantile function $\mathcal{Q}_{y_{it}}(\tau | \boldsymbol{z}_{t-1}), \tau \in (0,1)$, of a random variable y_{it} conditional on $\boldsymbol{z}_{t-1}$. For each $i = 1, \ldots, K$, it is identified through the following conditional quantile restriction

$$\Pr[y_{it} < \mathcal{Q}_{y_{it}}(\tau | \boldsymbol{z}_{t-1})] = \tau. \tag{1}$$

Yet, the data-generating process for $\boldsymbol{y}_t$ has to bear important characteristics. Namely, its vector conditional quantile function has to allow for not only the quantile specific intercept but also the slope effects. This way, the lagged information $\boldsymbol{z}_{t-1}$ and the shocks $\boldsymbol{\varepsilon}_t$ would have the state-specific impact on outcomes.[5] In what follows, for brevity in exposition I use the following notation $\mathcal{Q}_{\boldsymbol{y}_t}(\tau) = [\mathcal{Q}_{y_{1t}}(\tau), \ldots, \mathcal{Q}_{y_{Kt}}(\tau)]'$.

The following definition introduces a data-generating process for the random variables $\boldsymbol{y}_t$.[6]

[4] The mainstream empirical literature mainly concentrated on the modeling of the conditional mean function of $\boldsymbol{y}_t$ (see Lütkepohl (2005)).

[5] For comparison note that the linear VAR model can only produce a quantile specific intercept.

[6] This general form of a data-generating process can be viewed as a multivariate generalization of the model by Koenker and Xiao (2006). As it becomes evident later, additional modeling steps are to be taken for the definition and identification of the structural shocks.

Definition 1 (Random parameter VAR) *Let the nonlinear vector time series process* $\boldsymbol{y}_t$ *be expressed as a random parameter function of past values*

$$\boldsymbol{y}_t = \boldsymbol{\Phi}_{(0)t} + \sum_{j=1}^{p} \boldsymbol{\Phi}_{(j)t} \boldsymbol{y}_{t-j}, \tag{2}$$

where for i.i.d. standard uniform random variables $(u_{it}, i = 1, \ldots, K)$ *and the unknown real-valued function* $\left(\boldsymbol{\Phi}_{(j)}\right)_{ik}(\cdot)$ *the elements of the parameter matrices are defined as*

$$\left(\boldsymbol{\Phi}_{(j)t}\right)_{ik} = \left(\boldsymbol{\Phi}_{(j)}\right)_{ik}(u_{it}). \tag{3}$$

The time series process introduced in the definition 1 can incorporate various models with parameter instabilities. For instance, for some $\tilde{t} \in (0, T)$ and all $i, k \in (1, \ldots, K)$, let $\left(\boldsymbol{\Phi}_{(j)t}\right)_{ik} = \left(\boldsymbol{\Phi}_{(j)}\right)_{ik}(u_{i1})$ for $t \leq \tilde{t}$ and $\left(\boldsymbol{\Phi}_{(j)t}\right)_{ik} = \left(\boldsymbol{\Phi}_{(j)}\right)_{ik}(u_{i2})$ otherwise such that $u_{i1} \neq u_{i2}$, the process (2) becomes a threshold VAR. Alternatively, assume u_{it} is an exogenous latent factor with a probabilistic transition dynamics governed by a first order Markov chain. Then the model becomes a Markov switching VAR. In facts, the model displays ability to accommodate vast variety of time-series dynamics. Importantly, compared to the linear VAR models, developments in random component of the model would have implication both on the intercept and the slope coefficients.

Yet, for relating the process (3) to a conditional quantile function (1), I impose the following assumption.

Assumption 1 *For each* $i \in (1, \ldots, K)$*, the right hand side of* y_{it} *in the model* (2)-(3) *is monotone increasing in the i.i.d. standard uniform random variables* u_{it}*.*

Suppose, Assumption 1 holds and $\boldsymbol{y}_t$ follows the data-generating process in Definition 1. Then a $K \times 1$ vector of τ-th conditional quantile functions can be given as following

$$\mathcal{Q}_{\boldsymbol{y}_t}(\tau) = \boldsymbol{\Phi}_{(0)}(\tau) + \sum_{j=1}^{p} \boldsymbol{\Phi}_{(j)}(\tau) \boldsymbol{y}_{t-j}, \quad \tau \in (0, 1), \tag{4}$$

where the result is a consequence of the standard monotonic transformation $\mathcal{Q}_{y_{it}(u_{it})}(\tau) = y_{it}(\mathcal{Q}_{u_{it}}(\tau))$ for each $i = 1, 2, \ldots, K$. In this model coefficients are τ-dependent and therefore can have location, scale and shape effects on the distribution of a vector $\boldsymbol{y}_t$. Hence, the model implied by equations (2) and (4) is a multiple time-series version of the Koenker and Xiao (2006) model.

For analytic purposes it is convenient to rewrite equations (2) and (4) more concisely as following

$$\begin{aligned} \boldsymbol{z}_t &= \boldsymbol{\phi}_t + \boldsymbol{\Phi}_t \boldsymbol{z}_{t-1}, \\ \mathcal{Q}_{\boldsymbol{z}_t}(\tau) &= \boldsymbol{\phi}(\tau) + \boldsymbol{\Phi}(\tau)\boldsymbol{z}_{t-1}, \end{aligned} \tag{5}$$

where $\boldsymbol{z}_t = (\boldsymbol{y}_t', \boldsymbol{y}_{t-1}', \ldots, \boldsymbol{y}_{t-p+1}')'$, $\mathcal{Q}_{\boldsymbol{z}_t}(\tau) = (\mathcal{Q}_{\boldsymbol{y}_t}'(\tau), \boldsymbol{y}_{t-1}{}', \ldots, \boldsymbol{y}_{t-p+1}{}')'$ and parameter matrices are given as

$$\boldsymbol{\Phi}_t = \begin{bmatrix} \boldsymbol{\Phi}_{(1)t} & \boldsymbol{\Phi}_{(2)t} & \ldots & \boldsymbol{\Phi}_{(p-1)t} & \boldsymbol{\Phi}_{(p)t} \\ \boldsymbol{I}_K & \mathbf{O} & \ldots & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{I}_K & \ldots & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \ldots & \boldsymbol{I}_K & \mathbf{O} \end{bmatrix}, \quad \boldsymbol{\phi}_t = \begin{bmatrix} \boldsymbol{\Phi}_{(0)t} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$

$$\boldsymbol{\Phi}(\tau) = \begin{bmatrix} \boldsymbol{\Phi}_{(1)}(\tau) & \boldsymbol{\Phi}_{(2)}(\tau) & \ldots & \boldsymbol{\Phi}_{(p-1)}(\tau) & \boldsymbol{\Phi}_{(p)}(\tau) \\ \boldsymbol{I}_K & \mathbf{O} & \ldots & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{I}_K & \ldots & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \ldots & \boldsymbol{I}_K & \mathbf{O} \end{bmatrix}, \quad \boldsymbol{\phi}(\tau) = \begin{bmatrix} \boldsymbol{\Phi}_{(0)}(\tau) \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}.$$

The representation (5) is also known as the companion form of a VAR model. It is useful for relating $\mathcal{Q}_{\boldsymbol{z}_t}$ with own past realizations. In particular manipulating (5) results in the following proposition.

Proposition 1 (Vector quantile autoregression) *Suppose a $K \times 1$ vector of random variables $\boldsymbol{y}_t$ follows the model (3)-(2) with parameters satisfying Assumption 1. Then for a quantile $\tau \in (0,1)$, the vector quantile autoregression (VQAR) is defined as follows*

$$\mathcal{Q}_{\boldsymbol{z}_t}(\tau) = \boldsymbol{\phi}_{t-1}(\tau) + \boldsymbol{\Phi}_{t-1}(\tau)\mathcal{Q}_{\boldsymbol{z}_{t-1}}(\tau), \tag{6}$$

where the parameters are given as

$$\begin{aligned} \boldsymbol{\phi}_{t-1}(\tau) &= \boldsymbol{\phi}(\tau) + \boldsymbol{\Phi}(\tau)\boldsymbol{\phi}_{t-1} - \boldsymbol{\Phi}(\tau)\boldsymbol{\Phi}_{t-1}\boldsymbol{\Phi}(\tau)^{-1}\boldsymbol{\phi}(\tau) \\ \boldsymbol{\Phi}_{t-1}(\tau) &= \boldsymbol{\Phi}(\tau)\boldsymbol{\Phi}_{t-1}\boldsymbol{\Phi}(\tau)^{-1}. \end{aligned} \tag{7}$$

Proof 1 *See Appendix A.1.*

Proposition 1 is one of the building blocks of this paper, since it allows for tracing the dynamics of the shock transmission. Complication arises due to the fact that the parameters of the model now become functions of both the parameters of the data-generating process (2) and the parameters of the conditional quantile process (4). Therefore, behavior of the system (4) and its functionals at large depends on the specification of underlaying data-generating process. The following section discusses a framework for structural analysis by bridging a gap between shocks to the series $\boldsymbol{y}_t$ and its conditional quantile functions $\mathcal{Q}_{\boldsymbol{y}_t}(\tau)$.

3 Structural framework

Observed asymmetries in the most economic series could reflect events with a rare occurrence probability. I use probabilistic framework to define the structural shock and to assess its impact on the conditional distribution of outcomes.

3.1 Setup of the model

I assume that $\boldsymbol{y}_t$ is defined by the model (2), so that the outcomes ensure randomness through the parameter matrices $\boldsymbol{\Phi}_{(j)t}, j = 0, 1, \ldots, p, t \in \mathbb{Z}$. I view these matrices as affine transforms of a $K \times 1$ vector of the structural shocks $\boldsymbol{\xi}_t$ and detail them in the following definition.

Definition 2 (Parameter specification) *Let the model be given by 2, with the parameter matrices defined as*

$$\begin{aligned} \boldsymbol{\Phi}_{(j)t} &= \boldsymbol{\Phi}_{(j)} + \operatorname{diag}(\boldsymbol{\varepsilon}_t)\bar{\boldsymbol{\Phi}}_{(j)}, \quad j = 0, 1, \ldots, p, \\ \boldsymbol{\varepsilon}_t &\sim i.i.d.(\mathbf{0}_K, \boldsymbol{\Sigma}_K), \end{aligned} \tag{8}$$

where the parameters $\boldsymbol{\Phi}_{(j)}, \bar{\boldsymbol{\Phi}}_{(j)}, j = 0, 1, \ldots, p,$ and a positive definite matrix $\boldsymbol{\Sigma}_K$ are unknown.

The standard VAR is a model (8) with restrictions $\bar{\boldsymbol{\Phi}}_{(j)} = \mathbf{O}_K$ for $j = 1, 2, \ldots, p$. Therefore, the slope effect of shocks is ignored without a clear reason. For instance, there is no reason to expect that the relationship among fundamentals remain the same irrespective of the shocks and their size. Now, estimation of the entire parameter specification (8) becomes necessary for constructing the impulse response functions. It cannot be done using the ordinary least squares regression, because matrices $\bar{\boldsymbol{\Phi}}_{(j)}, j = 1, \ldots, p$ do not appear in the conditional mean specification. It cannot be directly done neither using the QR due to inseparability of the quantiles of $\boldsymbol{\varepsilon}_t$ and parameters $\bar{\boldsymbol{\Phi}}_{(j)}$s. To overcome this challenge, later in this section I develop the QR based iterative algorithm.

Yet an important question remains to be addressed. Namely, how the reduced-form shocks $\boldsymbol{\varepsilon}_t$ relate to each other. Here I propose the common identification strategy using the Cholesky recursive ordering.

Definition 3 (Structural shocks) *Let $\boldsymbol{P}$ be a $K \times K$ non-stochastic lower triangular matrix, $\boldsymbol{\xi}_t$ be a $K \times 1$ vector of cross- and -serially independent standardized random variables with unknown distribution functions $\mathrm{F}_{\xi_i}(\cdot), i = 1, \ldots, K$ and $\boldsymbol{u}_t = (u_{1t}, \ldots, u_{Kt})'$ be a $K \times 1$ vector of cross- and -serially independent standard uniform random variables. Then a $K \times 1$ vector of the reduced form shocks $\boldsymbol{\varepsilon}_t$ is defined as*

$$\begin{aligned} \boldsymbol{\varepsilon}_t &= \boldsymbol{P}\boldsymbol{\xi}_t, \\ &= \boldsymbol{P}\mathrm{F}^{-1}(\boldsymbol{u}_t), \end{aligned} \tag{9}$$

where the latter follows due to the inverse transform sampling.

The model summarized by equations (2), (8) and (9) defines the structural model, such that every nonzero realization of a shock contained in $\boldsymbol{\xi}_t$ will have the location, scale and shape effects on the conditional distribution of the outcomes $\boldsymbol{y}_t$.

As mentioned earlier, the parameters $\bar{\boldsymbol{\Phi}}_{(j)}, j = 1, \ldots, p$ can be directly estimated neither using the least squares nor the regression quantiles. However, it is still possible to calculate them using the QR estimates. To facilitate further discussions first note that the data-generating process (2) and (8) can be rewritten as

$$\boldsymbol{y}_t = \boldsymbol{\Phi}_{(0)} + \sum_{j=1}^{p} \boldsymbol{\Phi}_{(j)}\boldsymbol{y}_{t-j} + \mathrm{diag}\left(\bar{\boldsymbol{\Phi}}_{(0)} + \sum_{j=1}^{p} \bar{\boldsymbol{\Phi}}_{(j)}\boldsymbol{y}_{t-j}\right)\boldsymbol{\varepsilon}_t. \tag{10}$$

The conditional mean (median) estimates of the outcomes $\boldsymbol{y}_t$ will recover the parameters $\boldsymbol{\Phi}_j, j = 0, 1, \ldots, p$, whereas the conditional quantile estimates of the parameters for $\tau \neq 0.5$ will give

$$\boldsymbol{\Phi}_{(j)}(\tau) = \boldsymbol{\Phi}_{(j)} + \mathrm{diag}(\boldsymbol{\imath}_K \mathrm{F}^{-1}(\tau))\bar{\boldsymbol{\Phi}}_{(j)}, \quad j = 0, 1, \ldots, p, \tag{11}$$

where $\mathrm{F}(\cdot)$ is an unknown distribution function. Then,

$$\bar{\boldsymbol{\Phi}}_{(j)} = \left[\operatorname{diag}(\boldsymbol{\imath}_K \mathrm{F}^{-1}(\tau))\right]^{-1} \left(\boldsymbol{\Phi}_{(j)}(\tau) - \boldsymbol{\Phi}_{(j)}\right), \quad j = 0, 1, \ldots, p, \tag{12}$$

and

$$\boldsymbol{\varepsilon}_t = \left[\operatorname{diag}\left(\bar{\boldsymbol{\Phi}}_{(0)} + \sum_{j=1}^{p} \bar{\boldsymbol{\Phi}}_{(j)} \boldsymbol{y}_{t-j}\right)\right]^{-1} \left(\boldsymbol{y}_t - \boldsymbol{\Phi}_{(0)} - \sum_{j=1}^{p} \boldsymbol{\Phi}_{(j)} \boldsymbol{y}_{t-j}\right). \tag{13}$$

The following algorithm summarizes the iterative QR procedure:

(i) Estimate parameters of a conditional mean (median) and conditional quantile functions of observations $\boldsymbol{y}_t$.

(ii) Assume some parametric distribution function $\mathrm{F}(\cdot)$. For a quantile, $\tau \in (0, 1)$, $\tau \neq 0.5$, and the estimated parameters from the step (i), recover the remaining parameters and residuals using equations (12) and (13).

(iii) Construct the fitted values $\hat{\boldsymbol{y}}_t$ of $\boldsymbol{y}_t$ using equation (10) and evaluate the criteria $\frac{1}{T}\sum_{t=1}^{T} \psi\left(\hat{\boldsymbol{y}}_t - \boldsymbol{y}_t\right)$, where $\psi\left(u\right) = |u|^{\delta}$ for either $\delta = 1$ or $\delta = 2$.

(iv) Iterate over steps (ii) and (iii) for the finite grid of quantile values τ and select one with the lowest criteria.

Several alternative functions $\mathrm{F}(\cdot)$ could be considered based on their fitting performance.[7]

Standard impulse response analysis quantifies the extend and duration of an impact once fundamentals are drifted apart from the corresponding average path. In linear VARs the impulse responses at the shocks with different magnitudes differ only with a scale factor. Instead, in practice we observe that the probability at which a particular event happens matters. For example, the real economic activity decline due to a rarely observed stock market crush is important to analyze because these type of events have the long-lasting and devastating economy-wide effects.

To illustrate the modeling philosophy consider a following scenario. Suppose at time t, one of the variables y_{it} reaches its historically low level indexed by the probability τ, i.e

[7]Note that one could also impose the cross-quantile restriction (11) and evaluate simultaneously over the grid set of quantiles. However, this approach is computationally cumbersome and inflexible compared to one considered in this paper.

$y_{it} = \mathcal{Q}_{y_{it}}(\tau)$ such that $\Pr\left(y_{it} < \mathcal{Q}_{y_{it}}(\tau)\right) = \tau$. Then, due to the dependence (9) remaining fundamentals $y_{j\backslash i,t}$ will arrive at their new values implied by the tail event. This generates a hypothetical environment which is different from the one expected if none of such events would have happened. This difference constitutes our main interest, since it is fundamental for understanding implications of a variable y_{it} across its different states.

The following definition formalizes the foregone discussion.

Definition 4 (Quantile shock) *Let $\boldsymbol{y}_t$ be defined by equations (2), (8) and (9) and $\mathrm{F}_{\xi_i(\cdot)}$ be an empirical distribution function of $\boldsymbol{\xi}_t$. The quantile shock is defined as*

$$\xi_{it} = \mathrm{F}_{\xi_i}^{-1}(\tau), \quad \tau \in (0,1). \tag{14}$$

The value of $\boldsymbol{y}_t$ following the quantile shock $\boldsymbol{y}_t$ is referred to as the structural conditional quantile function

$$\mathcal{SQ}_{\boldsymbol{y}_t}(\tau) = \sum_{j=0}^{p} \left(\boldsymbol{\Phi}_{(j)} + \mathrm{diag}\left(\boldsymbol{P}\tilde{\boldsymbol{\imath}}_K \mathrm{F}^{-1}(\tau)\right) \bar{\boldsymbol{\Phi}}_{(j)}\right) \boldsymbol{y}_{t-j} \tag{15}$$

where $\tilde{\boldsymbol{\imath}}_K$ is a $K \times 1$ vector with one as i-th element and zeros elsewhere and $\boldsymbol{P}$ is given the definition 3.

The quantile shock to ξ_{it} sets the value of y_{it} to its conditional quantile function $\mathcal{Q}_{y_{it}}(\tau)$, whereas other elements in $\mathcal{SQ}_{\boldsymbol{y}_t}(\tau)$ will be implied by the dependence structure $\boldsymbol{P}$.

4 Impulse response functions

I introduce the quantile impulse response functions (QIRFs) based on the notion of the quantile shocks as introduced in the definition 4. The following definition formalizes the concept of the QIRFs for the conditional vector quantile process in the equation (4).

Definition 5 (QIRFs) *Let the results of Proposition 1 hold. Then, for the information set $\mathcal{I}_{t-1} \equiv (\boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-1}, \dots)$ and the definitions 3 and 4, the quantile impulse response functions are given as*

$$\boldsymbol{\Delta}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau) \;\;=\;\; \mathrm{E}\left[\mathcal{Q}_{\boldsymbol{z}_{t+h}}(\tau) | \xi_{it} = \mathrm{F}_{\xi_i}^{-1}(\tau), \mathcal{I}_{t-1}\right] - \mathrm{E}\left[\mathcal{Q}_{\boldsymbol{z}_{t+h}}(\tau) | \mathcal{I}_{t-1}\right], \tag{16}$$

where $h \geq 1$ is the horizon of an impact and $F_{\xi_i}(\cdot)$ is an empirical distribution function.

There are several important properties of the QIRFs to be highlighted. The QIRFs can be interpreted as the difference between the expected conditional quantile function and its counterfactual under the quantile shock scenario. Since, using the methodology developed in this paper the counterfactual distribution can be easily constructed for any other scenario such are the simultaneous or sequential multiple quantile shocks for various stress testing exercises.

I impose the standard stability condition.

Assumption 2 *Eigenvalues of the matrix $\tilde{\boldsymbol{\Phi}}(\tau) = \mathrm{E}\,\boldsymbol{\Phi}_t(\tau)$ are less than one in absolute value.*

The following theorem establishes the main results of the paper.

Theorem 1 (Functional form of QIRFs) *Suppose Assumptions 1 and 2 hold. Then, for the model implied by Proposition 1, the QIRFs introduced in Definition 5 can be expressed as*

$$\boldsymbol{\Delta}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau) \;=\; \bar{\boldsymbol{\Phi}}(\tau)^{h-1}\left(\left[\tilde{\boldsymbol{\phi}}(\tau) - \bar{\boldsymbol{\phi}}(\tau)\right] + \left[\tilde{\boldsymbol{\Phi}}(\tau) - \bar{\boldsymbol{\Phi}}(\tau)\right]\mathcal{Q}_{\boldsymbol{z}_t}(\tau)\right), \quad h = 1, 2, \ldots, \tag{17}$$

where $\tilde{\boldsymbol{\phi}}(\tau) \equiv \mathrm{E}[\boldsymbol{\phi}_t(\tau)|\xi_{it} = F^{-1}_{\xi_i}(\tau), \mathcal{I}_{t-1}]$, $\tilde{\boldsymbol{\Phi}}(\tau) \equiv \mathrm{E}[\boldsymbol{\Phi}_t(\tau)|\xi_{it} = F^{-1}_{\xi_i}(\tau), \mathcal{I}_{t-1}]$, $\bar{\boldsymbol{\Phi}}(\tau) \equiv \mathrm{E}[\boldsymbol{\Phi}_t(\tau)|\mathcal{I}_{t-1}]$ *and* $\bar{\boldsymbol{\phi}}(\tau) \equiv \mathrm{E}[\boldsymbol{\phi}_t(\tau)|\mathcal{I}_{t-1}]$.

Proof 2 *See Appendix A.2.*

Suppose the parameters $\left\{\hat{\boldsymbol{\Phi}}_{(j)}, \hat{\bar{\boldsymbol{\Phi}}}_{(j)}, \hat{\boldsymbol{\Phi}}_{(j)}(\tau)\right\}_{j=0}^{p}$, $\hat{\boldsymbol{P}}$ and the vector of quantile functions $\hat{\mathcal{Q}}_{\boldsymbol{z}_t}(\tau)$ are available. Then, estimates of the QIRFs can be given as the upper $K \times 1$ block of the following vector

$$\hat{\boldsymbol{\Delta}}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau) \;=\; \hat{\bar{\boldsymbol{\Phi}}}(\tau)^{h-1}\left(\left[\hat{\tilde{\boldsymbol{\phi}}}(\tau) - \hat{\bar{\boldsymbol{\phi}}}(\tau)\right] + \left[\hat{\tilde{\boldsymbol{\Phi}}}(\tau) - \hat{\bar{\boldsymbol{\Phi}}}(\tau)\right]\hat{\mathcal{Q}}_{\boldsymbol{z}_t}(\tau)\right), \quad h = 1, 2, \ldots, \tag{18}$$

with the parameter matrices defined as

$$\begin{aligned}
\hat{\bar{\boldsymbol{\Phi}}}(\tau) &= \hat{\boldsymbol{\Phi}}(\tau)\hat{\bar{\boldsymbol{\Phi}}}\hat{\boldsymbol{\Phi}}(\tau)^{-1} \\
\hat{\tilde{\boldsymbol{\Phi}}}(\tau) &= \hat{\boldsymbol{\Phi}}(\tau)\hat{\tilde{\boldsymbol{\Phi}}}\hat{\boldsymbol{\Phi}}(\tau)^{-1} \\
\hat{\bar{\boldsymbol{\phi}}}(\tau) &= \hat{\boldsymbol{\phi}}(\tau) + \hat{\boldsymbol{\Phi}}(\tau)\hat{\bar{\boldsymbol{\phi}}} - \hat{\boldsymbol{\Phi}}(\tau)\hat{\bar{\boldsymbol{\Phi}}}\hat{\boldsymbol{\Phi}}(\tau)^{-1}\hat{\boldsymbol{\phi}}(\tau) \\
\hat{\tilde{\boldsymbol{\phi}}}(\tau) &= \hat{\boldsymbol{\phi}}(\tau) + \hat{\boldsymbol{\Phi}}(\tau)\hat{\tilde{\boldsymbol{\phi}}} - \hat{\boldsymbol{\Phi}}(\tau)\hat{\tilde{\boldsymbol{\Phi}}}\hat{\boldsymbol{\Phi}}(\tau)^{-1}\hat{\boldsymbol{\phi}}(\tau),
\end{aligned} \tag{19}$$

where

$$\hat{\boldsymbol{\Phi}} = \begin{bmatrix} \hat{\boldsymbol{\Phi}}_{(1)} & \hat{\boldsymbol{\Phi}}_{(2)} & \dots & \hat{\boldsymbol{\Phi}}_{(p-1)} & \hat{\boldsymbol{\Phi}}_{(p)} \\ \boldsymbol{I}_K & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{I}_K & \dots & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & \boldsymbol{I}_K & \mathbf{O} \end{bmatrix}, \quad \hat{\boldsymbol{\phi}} = \begin{bmatrix} \hat{\boldsymbol{\Phi}}_{(0)} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$

$$\hat{\boldsymbol{\Phi}}(\tau) = \begin{bmatrix} \hat{\boldsymbol{\Phi}}_{(1)}(\tau) & \hat{\boldsymbol{\Phi}}_{(2)}(\tau) & \dots & \hat{\boldsymbol{\Phi}}_{(p-1)}(\tau) & \hat{\boldsymbol{\Phi}}_{(p)}(\tau) \\ \boldsymbol{I}_K & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{I}_K & \dots & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & \boldsymbol{I}_K & \mathbf{O} \end{bmatrix}, \quad \hat{\boldsymbol{\phi}}(\tau) = \begin{bmatrix} \hat{\boldsymbol{\Phi}}_{(0)}(\tau) \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}.$$

Furthermore, let $\tilde{\boldsymbol{\imath}}_K$ be $K \times 1$ vector with one as ith element and with zeros elsewhere and let the expected values of the time-varying parameters conditional on shock be defined as

$$\hat{\tilde{\boldsymbol{\Phi}}}_{(j)} = \hat{\boldsymbol{\Phi}}_{(j)} + \operatorname{diag}(\hat{\boldsymbol{P}}\tilde{\boldsymbol{\imath}}_K \mathrm{F}^{-1}(\tau))\hat{\bar{\boldsymbol{\Phi}}}_{(j)}, \quad j = 0, 1, \dots, p. \tag{20}$$

Then the remaining components of the QIRFs can be given as

$$\hat{\tilde{\boldsymbol{\Phi}}} = \begin{bmatrix} \hat{\tilde{\boldsymbol{\Phi}}}_{(1)} & \hat{\tilde{\boldsymbol{\Phi}}}_{(2)} & \dots & \hat{\tilde{\boldsymbol{\Phi}}}_{(p-1)} & \hat{\tilde{\boldsymbol{\Phi}}}_{(p)} \\ \boldsymbol{I}_K & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{I}_K & \dots & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & \boldsymbol{I}_K & \mathbf{O} \end{bmatrix}, \quad \hat{\tilde{\boldsymbol{\phi}}} = \begin{bmatrix} \hat{\tilde{\boldsymbol{\Phi}}}_{(0)} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}.$$

Due to its popularity, the VAR model with constant parameters is of particular interest. I establish the following corollary to give QIRFs for those type of models.

Corollary 1 (VAR with constant parameters) *Consider the following linear parameter model with Gaussian innovations*

$$\boldsymbol{y}_t = \boldsymbol{\Phi}_{(0)t} + \textstyle\sum_{j=1}^{p} \boldsymbol{\Phi}_{(j)}\boldsymbol{y}_{t-j}, \tag{21}$$

where $\boldsymbol{\Phi}_{(0)t} = \boldsymbol{\Phi}_{(0)} + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \sim i.i.d.(\mathbf{0}, \boldsymbol{\Sigma})$ *and suppose the relationship (9) holds. Then the QIRFs can be given as*

$$\boldsymbol{\Delta}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau) = \boldsymbol{\Phi}^{h-1}\left(\tilde{\boldsymbol{\phi}}(\tau) - \bar{\boldsymbol{\phi}}(\tau)\right), \quad h = 1, 2, \dots.$$

where $\tilde{\boldsymbol{\imath}}_K$ *is a vector with one as i-th element and zeros elsewhere,* $\tilde{\boldsymbol{\phi}}(\tau) \equiv \mathrm{E}[\boldsymbol{\phi}_t(\tau)|\xi_{it} = F_{\xi_i}^{-1}(\tau), \mathcal{I}_{t-1}]$ *and* $\bar{\boldsymbol{\phi}}(\tau) \equiv \mathrm{E}[\boldsymbol{\phi}_t(\tau)|\mathcal{I}_{t-1}]$.

There are several findings worth to mention. First, note that this model does not make quantile regression operational, since the parameter matrices $\boldsymbol{\Phi}$ and $\boldsymbol{P}$ can be estimated using least squares regression. Now let for some $\mathrm{F}_{\xi_i}^{-1}(\tau) = 1$ for some $\tau \in (0,1)$, then the impulse responses will take familiar form from the linear regression literature. Second, if the data-generating process (2) is not supported by the data then these two impulse response functions must not be statistically different from each other. I use this fact for evaluating the specification of the underlying data.

5 Estimation and inference

This section contains estimation and theoretical results of the paper, necessary for the QIRF analysis. In particular, for ease of inference, I discuss the bootstrap algorithm for construction of confidence bands.

First note that, for $i = 1, \ldots, K$, estimates of the parameters satisfy the following equations

$$\left\{ \left(\hat{\boldsymbol{\Phi}}_{(j)} \right)_{i \cdot} \right\}_{j=0}^{p} = \underset{\left\{ \left(\boldsymbol{\Phi}_{(j)} \right)_{i \cdot} \right\}_{j=0}^{p} \in \mathbb{R}^{Kp+1}}{\arg\min} \sum_{t=1}^{T} \left| y_{it} - \left(\boldsymbol{\Phi}_{(0)} \right)_{i \cdot} + \sum_{j=1}^{p} \left(\boldsymbol{\Phi}_{(j)} \right)_{i \cdot} \boldsymbol{y}_{t-j} \right|, \tag{22}$$

and

$$\left\{ \left(\hat{\boldsymbol{\Phi}}_{(j)}(\tau) \right)_{i \cdot} \right\}_{j=0}^{p} = \underset{\left\{ \left(\boldsymbol{\Phi}_{(j)}(\tau) \right)_{i \cdot} \right\}_{j=0}^{p} \in \mathbb{R}^{K(p+1)}}{\arg\min} \sum_{t=1}^{T} \rho_\tau \left(y_{it} - \left(\boldsymbol{\Phi}_{(0)}(\tau) \right)_{i \cdot} + \sum_{j=1}^{p} \left(\boldsymbol{\Phi}_{(j)}(\tau) \right)_{i \cdot} \boldsymbol{y}_{t-j} \right), \tag{23}$$

where $\rho_\tau(u) = u\left(\tau - I(u < 0)\right)$ is the asymmetric loss function by Koenker and Bassett (1978).[8]

Algorithm in the section 3 can be applied to estimate the sequence of parameter matrices $\left\{ \hat{\bar{\boldsymbol{\Phi}}}_{(j)} \right\}_{j=0}^{p}$, the vector of residuals $\hat{\boldsymbol{\varepsilon}}_t$ and its empirical covariance function $\hat{\boldsymbol{\Sigma}}$ with $\hat{\boldsymbol{P}}$ as the

[8]Note that the sequence of parameter vectors $\left\{ \left(\hat{\boldsymbol{\Phi}}_{(j)} \right)_{i \cdot} \right\}_{j=0}^{p}$ for $i = 1, \ldots, K$ can also be estimated using ordinary least squares regession. However, the median regression outcome (22) has various advantages over the least squares approach including robustness to outlines.

lower Cholesky factor. Then, the QIRFs can be constructed by following the discussion in Section (4).

Now let $\hat{\boldsymbol{\theta}} = \text{vec}\left[\hat{\boldsymbol{\Phi}}_{(0)}, \ldots, \hat{\boldsymbol{\Phi}}_{(p)}, \hat{\bar{\boldsymbol{\Phi}}}_{(0)}, \ldots, \hat{\bar{\boldsymbol{\Phi}}}_{(p)}, \hat{\boldsymbol{\Phi}}_{(0)}(\tau), \ldots, \hat{\boldsymbol{\Phi}}_{(p)}(\tau), \hat{\boldsymbol{\Sigma}}\right]$ be the vector of parameter estimates. Following arguments of Koenker and Xiao (2006) and Section 6 in Newey and McFadden (1994) it is straightforward to conclude that as $T \to \infty$ the vector of parameters satisfy $\sqrt{T}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) \sim \mathrm{N}\left(\mathbf{0}, \boldsymbol{\Omega}_{\boldsymbol{\theta}}\right)$, where $\boldsymbol{\Omega}_{\boldsymbol{\theta}}$ is an asymptotic covariance matrix. Then, using results by Serfling (1980, p.122) the QIRFs satisfy $\sqrt{T}\left(\hat{\boldsymbol{\Delta}}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau) - \boldsymbol{\Delta}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau)\right) \sim \mathrm{N}\left(\mathbf{0}, \nabla_{\boldsymbol{\theta}'}\boldsymbol{\Delta}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau)\boldsymbol{\Omega}_{\boldsymbol{\theta}}\nabla_{\boldsymbol{\theta}}\boldsymbol{\Delta}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau)'\right)$. The limiting result can be characterized either analytically or approximated numerically. However, this turns out to be a challenging task. Therefore, for ease of inference as well as for reliable small sample performance I propose the following standard bootstrap algorithm.

Algorithm 1 (Bootstrap for QIRFs) *(i) First, obtain estimates of the model parameters and residuals* $(\hat{\boldsymbol{\varepsilon}}_t)_{t=1}^T$ *using equations* (22) *and* (23) *and the algorithm in the section 3, then construct* $\hat{\boldsymbol{\Delta}}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau)$ *for some* $\tau \in (0,1)$ *and* $h \geq 1$ *as discussed in Section 4. (ii) Generate bootstrap sample* $\left(\hat{\boldsymbol{\varepsilon}}_t^{(b)}\right)_{t=1}^T$ *by randomly drawing without replacement from* $(\hat{\boldsymbol{\varepsilon}}_t)_{t=1}^T$ *and conditionally on* $(\boldsymbol{y}_p', \boldsymbol{y}_{p-1}', \ldots, \boldsymbol{y}_1')'$ *recursively construct time series* $\left(\boldsymbol{y}_t^{(b)}\right)_{t=1}^T$ *using equation* (10)*. (iii) Reestimate the model and calculate bootstrap estimates of the QIRFs* $\hat{\boldsymbol{\Delta}}^{(b)}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau)$*. (iv) Repeat steps (ii) - (iii) for a sufficiently large number of times and construct bootstrap estimates of the asymptotic covariance matrix using empirical covariance of the bootstrap sample* $\hat{\boldsymbol{\Delta}}^{(1)}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau), \ldots, \hat{\boldsymbol{\Delta}}^{(B)}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau)$.[9]

Algorithm (1) can be used to construct functional hypothesis test on the QIRFs.

6 Finite Sample Assessments

Although the estimation approach considered in the previous section is fairly intuitive, I run the series of Monte Carlo specifications to asses validity of the estimation approach.

[9]Bootstrap technique has been widely applied in quantile regression framework previously, e.g., Hahn (1995, 1997).

6.1 Simulation Design

For brevity in demonstration I consider a bivariate setting. Throughout this exercise the following coefficient matrices are considered:

$$\boldsymbol{\Phi}_{(0)} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \boldsymbol{\Phi}_{(1)} = \begin{bmatrix} 0.7 & 0.2 \\ 0.1 & 0.55 \end{bmatrix},$$

$$\bar{\boldsymbol{\Phi}}_{(0)} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \bar{\boldsymbol{\Phi}}_{(1)} = \begin{bmatrix} 0.2 & 0.1 \\ 0.05 & 0.3 \end{bmatrix}, \boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

where $\rho = 0.9$. Data is generated recursively using the process (10) where the vector $\boldsymbol{\varepsilon}_t$ is assumed to follow standardized Student's t and Chi-squared distributions with 3 degress of freedom. I set $T = (250, 550, 1050)$ while discarding first 50 observations and carry 1000 replications, computing the empirical mean and variance of the QIRFs as well as its true values.

[**Include Figures 1 and 2 around here.**]

6.2 Results

As a first experiment, I generate time series of $\boldsymbol{y}_t$ characterized by asymmetries in the tail behavior as shown in Figures 1 and 2. Several results require comments. Asymmetries depicted in the data relative to the symmetric normal distribution are mirrored in the QIRFs. In particular the quantile shocks at $\tau = 0.9$ has higher impact on the right tail whereas its left tail counterpart at $\tau = 0.1$. As we move closer to the median the data gets closer to the normal distribution and the QIRFs become more symmetric as show in Figure 2. This supports the thesis that asymmetries in some macroeconomic data could be well explained by shocks that occur with different probabilities.

[**Include Figures 3 to 10 around here.**]

The finite sample property of estimates are reported in Figures 3-6 for Student's t distributed data and Figures 7-10 for Chi-Squared distributed data. The true QIRFs are taken as an average over the number of simulated samples. For the estimation Algorithm 1, I use standard normal distribution function as a true one. The estimation results are represented as an empirical mean as well as two standard deviation confidence bands approximated by the ratio of the empirical to the standard normal interquartile ranges. There are several remarks in order. First, irrespective of the data distribution as well as sample size estimates of the QIRFs fall close to the true counterparts. Second, the empirical two standard deviation bands get narrower with the sample size.

7 Conclusions

Gaining a uniform view on covariate effects is in the heart of empirical analysis. In this paper, I have proposed a method that allows understanding impact and a transmission of a shock that occur with a certain regularity. Furthermore, as I argue in this paper, this tool might be an effective design in explaining asymmetric behavior of many time-series fundamentals and can be used as an effective tool in economic stress testing exercises.

Appendices

A Proofs

This section contains extended discussions on the main statistical results of the paper.

A.1 Proof of Proposition 1

First note that the companion form of the model in Equation (5) can be written as

$$\begin{aligned} \boldsymbol{z}_{t-1} &= \boldsymbol{\Phi}_t^{-1}\left(\boldsymbol{z}_t - \boldsymbol{\phi}_t\right), \\ \boldsymbol{z}_{t-1} &= \boldsymbol{\Phi}(\tau)^{-1}\left[\mathcal{Q}_{\boldsymbol{z}_t}(\tau) - \boldsymbol{\phi}(\tau)\right], \end{aligned}$$

which results in the following equation

$$\boldsymbol{z}_t = \boldsymbol{\phi}_t + \boldsymbol{\Phi}_t \boldsymbol{\Phi}(\tau)^{-1}\left[\mathcal{Q}_{\boldsymbol{z}_t}(\tau) - \boldsymbol{\phi}(\tau)\right]. \tag{24}$$

Then the result follows by plugging (24) into (5)

$$\begin{aligned} \mathcal{Q}_{\boldsymbol{z}_t}(\tau) &= \boldsymbol{\phi}(\tau) + \boldsymbol{\Phi}(\tau)(\boldsymbol{\phi}_{t-1} + \boldsymbol{\Phi}_{t-1}\boldsymbol{\Phi}(\tau)^{-1}[\mathcal{Q}_{\boldsymbol{z}_{t-1}}(\tau) - \boldsymbol{\phi}(\tau)]) \\ &= [\boldsymbol{\phi}(\tau) + \boldsymbol{\Phi}(\tau)\boldsymbol{\phi}_{t-1} - \boldsymbol{\Phi}(\tau)\boldsymbol{\Phi}_{t-1}\boldsymbol{\Phi}(\tau)^{-1}\boldsymbol{\phi}(\tau)] + \boldsymbol{\Phi}(\tau)\boldsymbol{\Phi}_{t-1}\boldsymbol{\Phi}(\tau)^{-1}\mathcal{Q}_{\boldsymbol{z}_{t-1}}(\tau) \\ &= \boldsymbol{\phi}_{t-1}(\tau) + \boldsymbol{\Phi}_{t-1}(\tau)\mathcal{Q}_{\boldsymbol{z}_{t-1}}(\tau). \end{aligned}$$

A.2 Proof of the Theorem 1

Note that

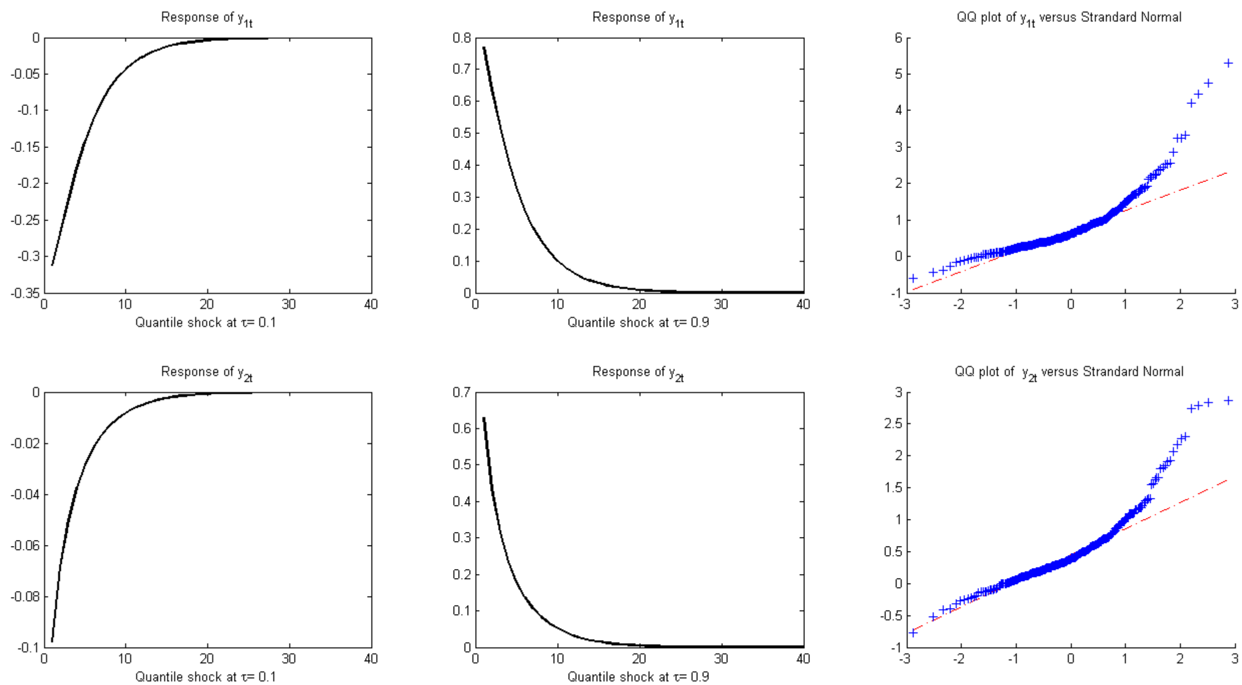
$$\mathcal{Q}_{\boldsymbol{z}_{t+h}}(\tau) = \boldsymbol{\phi}_{t+h-1}(\tau) + \sum_{l=2}^{h-1}\prod_{j=1}^{l-1}\boldsymbol{\Phi}_{t+h-j}(\tau)\boldsymbol{\phi}_{t+h-l}(\tau) + \prod_{j=1}^{h-1}\Phi_{t+h-j}(\tau)\left[\phi_t(\tau) + \boldsymbol{\Phi}_t(\tau)\mathcal{Q}_{\boldsymbol{z}_t}(\tau)\right].$$

Since the vector of random variables $\boldsymbol{\varepsilon}_t$ is i.i.d. the result immediately follows as

$$\begin{aligned} \boldsymbol{\Delta}_{\mathcal{Q}_{\boldsymbol{z}_{t+h}}}(\tau) &= \mathrm{E}\left[\mathcal{Q}_{\boldsymbol{z}_{t+h}}(\tau)|\xi_{it} = F^{-1}(\tau), \mathcal{I}_{t-1}\right] - \mathrm{E}\left[\mathcal{Q}_{\boldsymbol{z}_{t+h}}|\mathcal{I}_{t-1}\right] \\ &= \bar{\boldsymbol{\Phi}}(\tau)^{h-1}\left(\left[\tilde{\boldsymbol{\phi}}(\tau) - \bar{\boldsymbol{\phi}}(\tau)\right] + \left[\tilde{\boldsymbol{\Phi}}(\tau) - \bar{\boldsymbol{\Phi}}(\tau)\right]\mathcal{Q}_{\boldsymbol{z}_t}(\tau)\right), \quad h = 1, 2, \ldots, \end{aligned} \tag{25}$$

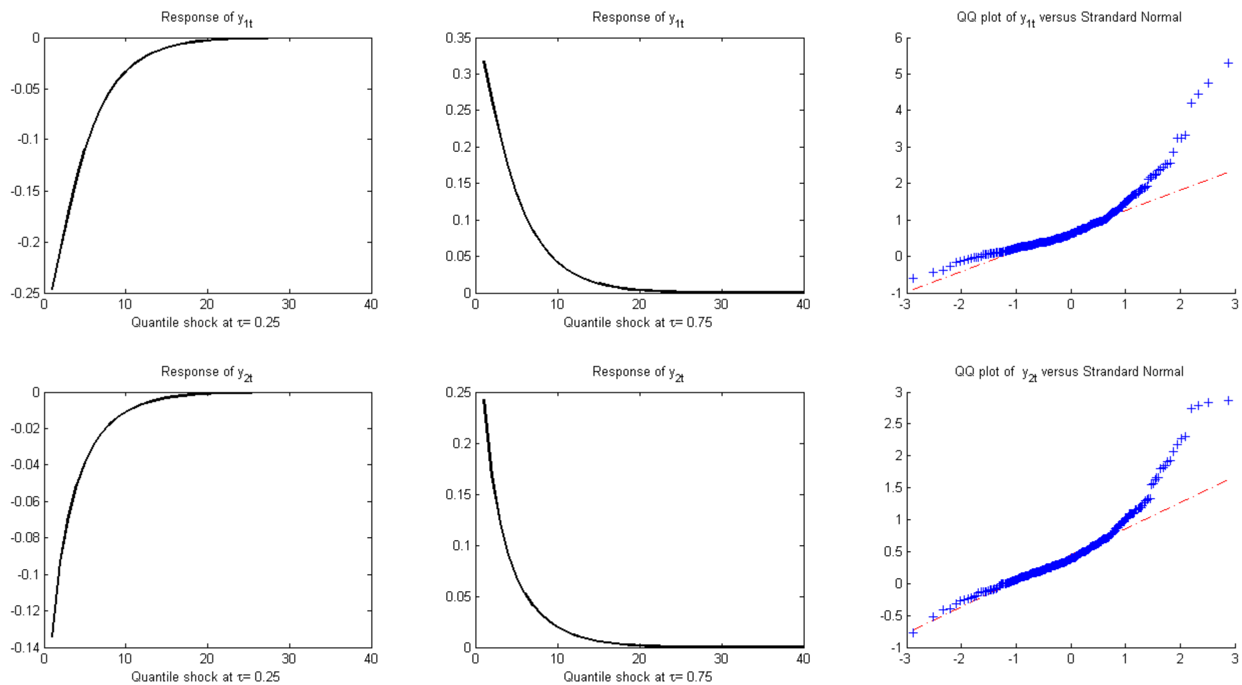
where $\tilde{\boldsymbol{\phi}}(\tau) \equiv \mathrm{E}[\boldsymbol{\phi}_t(\tau)|\xi_{it} = F^{-1}(\tau), \mathcal{I}_{t-1}]$, $\tilde{\boldsymbol{\Phi}}(\tau) \equiv \mathrm{E}[\boldsymbol{\Phi}_t(\tau)|\xi_{it} = F^{-1}(\tau), \mathcal{I}_{t-1}]$, $\bar{\boldsymbol{\Phi}}(\tau) \equiv \mathrm{E}[\boldsymbol{\Phi}_t(\tau)|\mathcal{I}_{t-1}]$ and $\mathrm{E}\left[\boldsymbol{\phi}_t(\tau)|\mathcal{I}_{t-1}\right]$.

Figure 1: Impulse Responses to a Shock in y_{1t}.



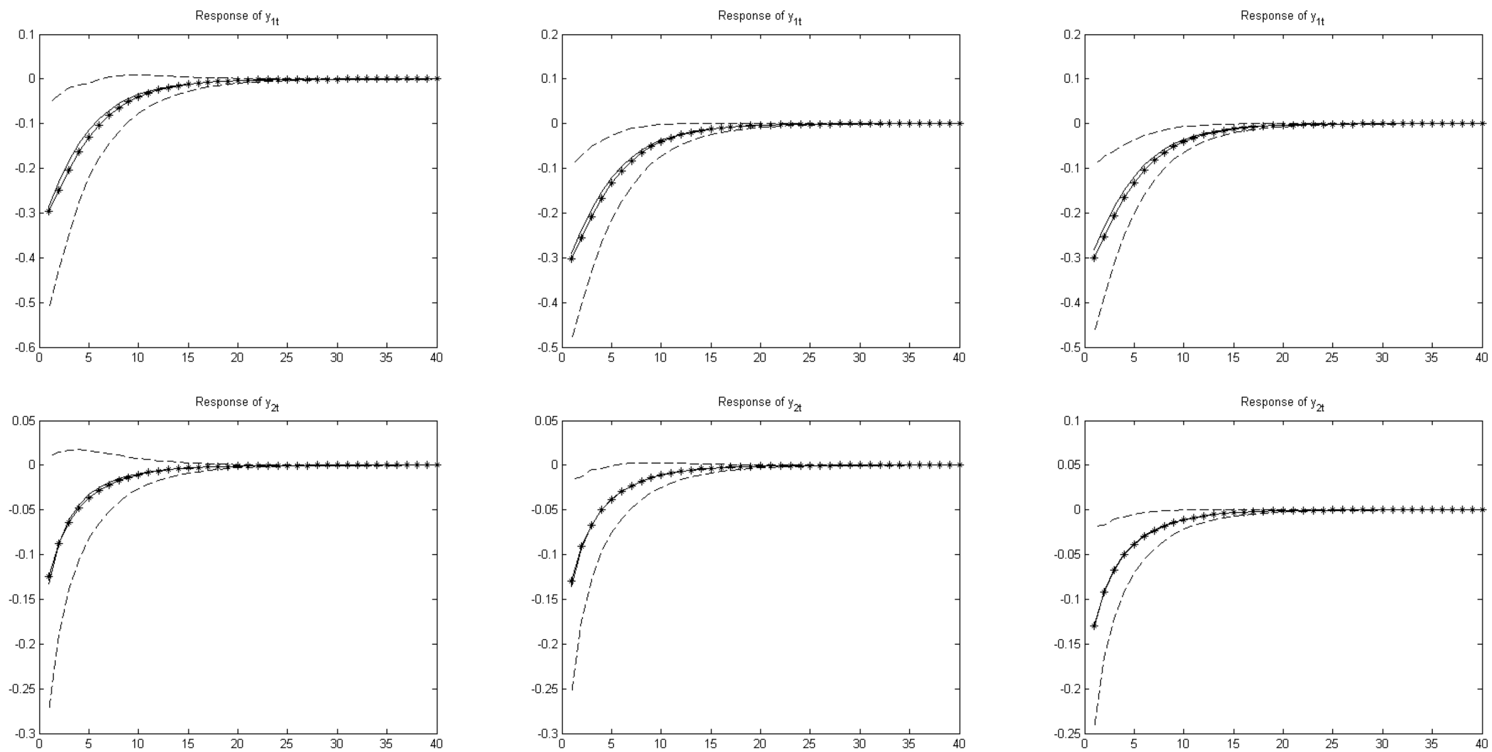
Note: True impulse response functions, based on a bivariate VAR(1) model, follow the design in Section (6.1). Two impulse responses are compared with asymmetric data behavior for quantile values $\tau, 1-\tau$ with $\tau = 0.1$ depicted in the quantile-to-quantile plots.

Figure 2: Impulse Responses to a Shock in y_{1t}.



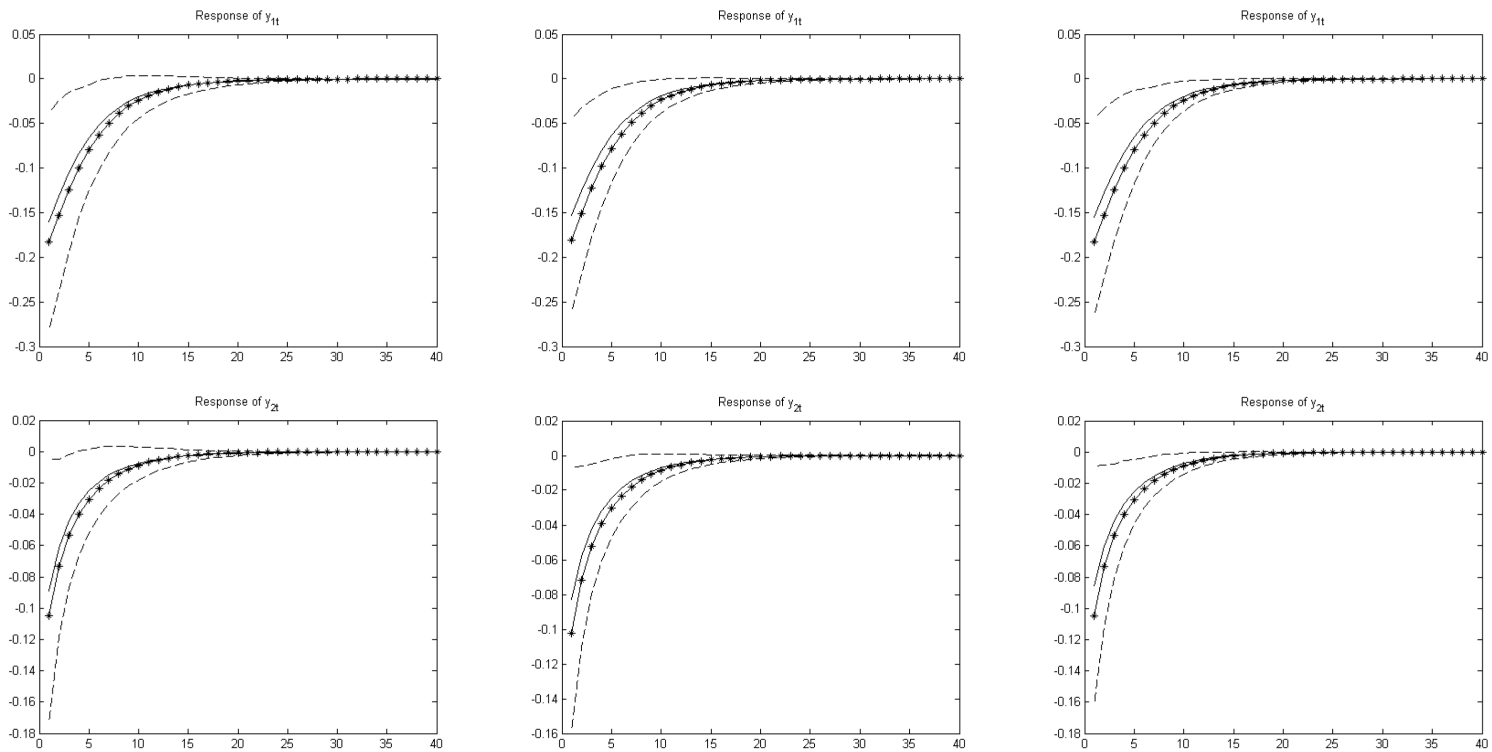
Note: True impulse response functions, based on a bivariate VAR(1) model, follow the design in Section (6.1). Two impulse responses are compared with asymmetric data behavior for quantile values $\tau, 1-\tau$ with $\tau = 0.25$ depicted in the quantile-to-quantile plots.

Figure 3: Estimated Impulse Responses to a Shock in y_{1t} for $\tau = 0.1$.



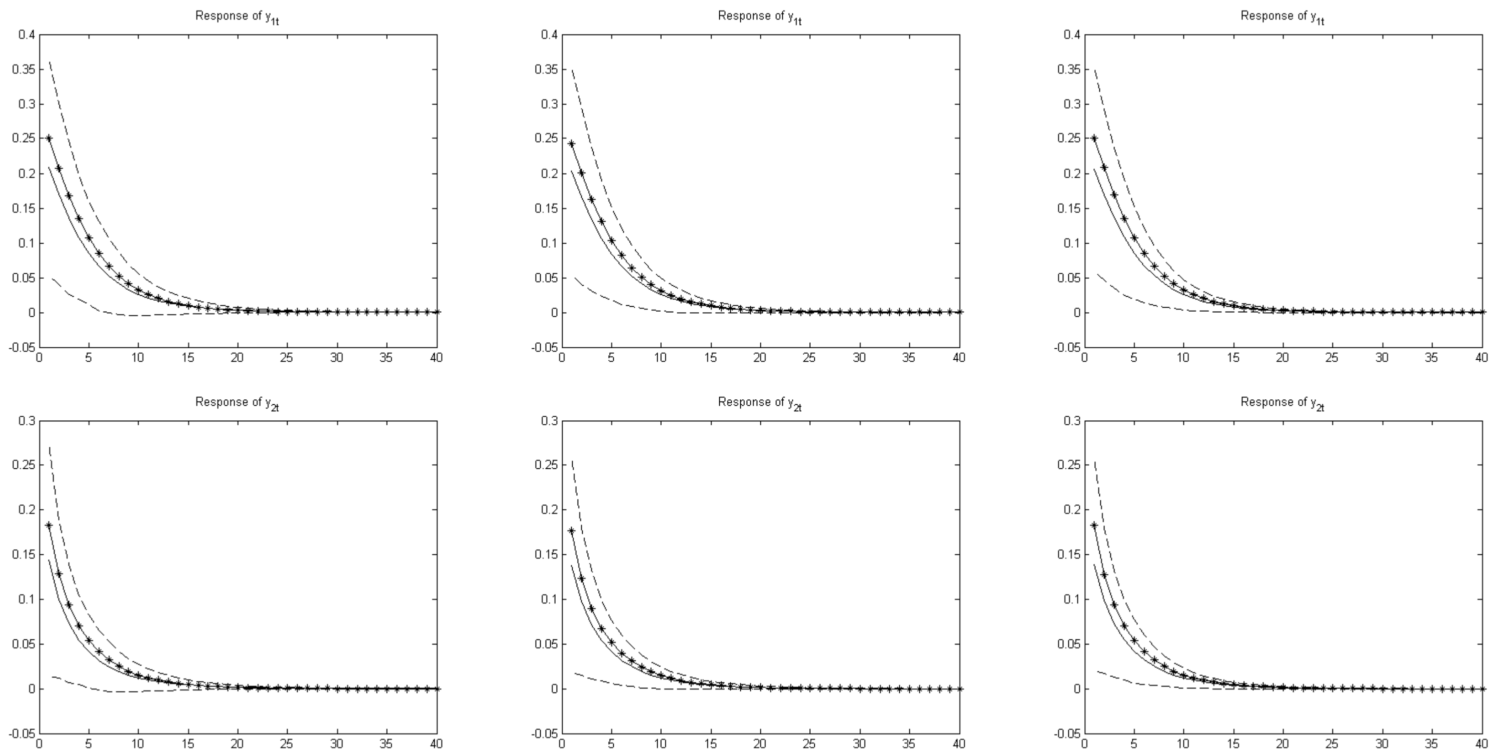
Note: True impulse response functions (–*) based on a random variable bivariate VAR(1) model following the design in Section (6.1) are compared with estimated ones (–*) with two standard-error bands (- -) calculated by empirical interquartile range scaled by the one calculated using normal distribution. The quantile value is set to $\tau = 0.1$. Data is generated using standardized Student's t distribution with 3 degrees of freedom.

Figure 4: Estimated Impulse Responses to a Shock in y_{1t} for $\tau = 0.25$.



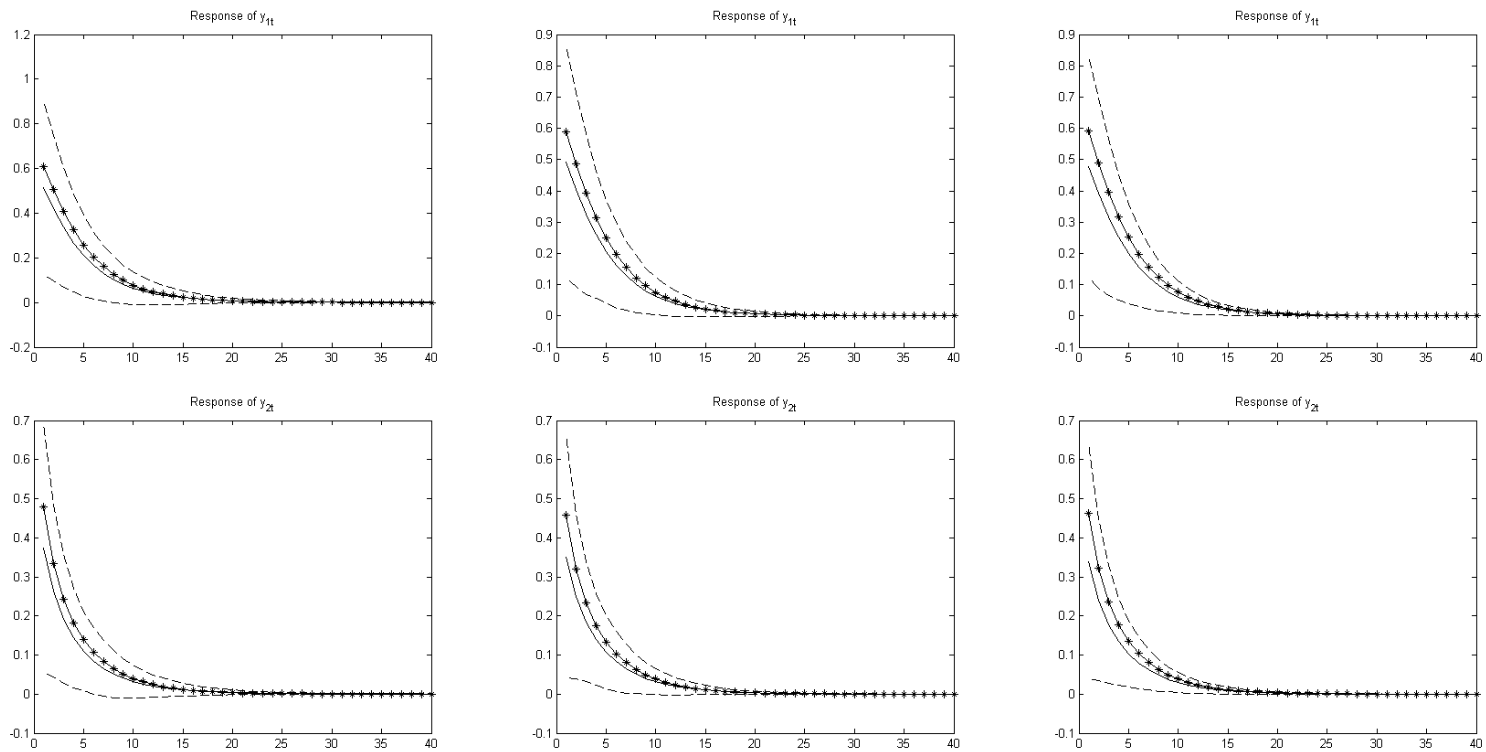
Note: True impulse response functions (–*) based on a random variable bivariate VAR(1) model following the design in Section (6.1) are compared with estimated ones (–*) with two standard-error bands (- -) calculated by empirical interquartile range scaled by the one calculated using normal distribution. The quantile value is set to $\tau = 0.25$. Data is generated using standardized Student's t distribution with 3 degrees of freedom.

Figure 5: Estimated Impulse Responses to a Shock in y_{1t} for $\tau = 0.75$.



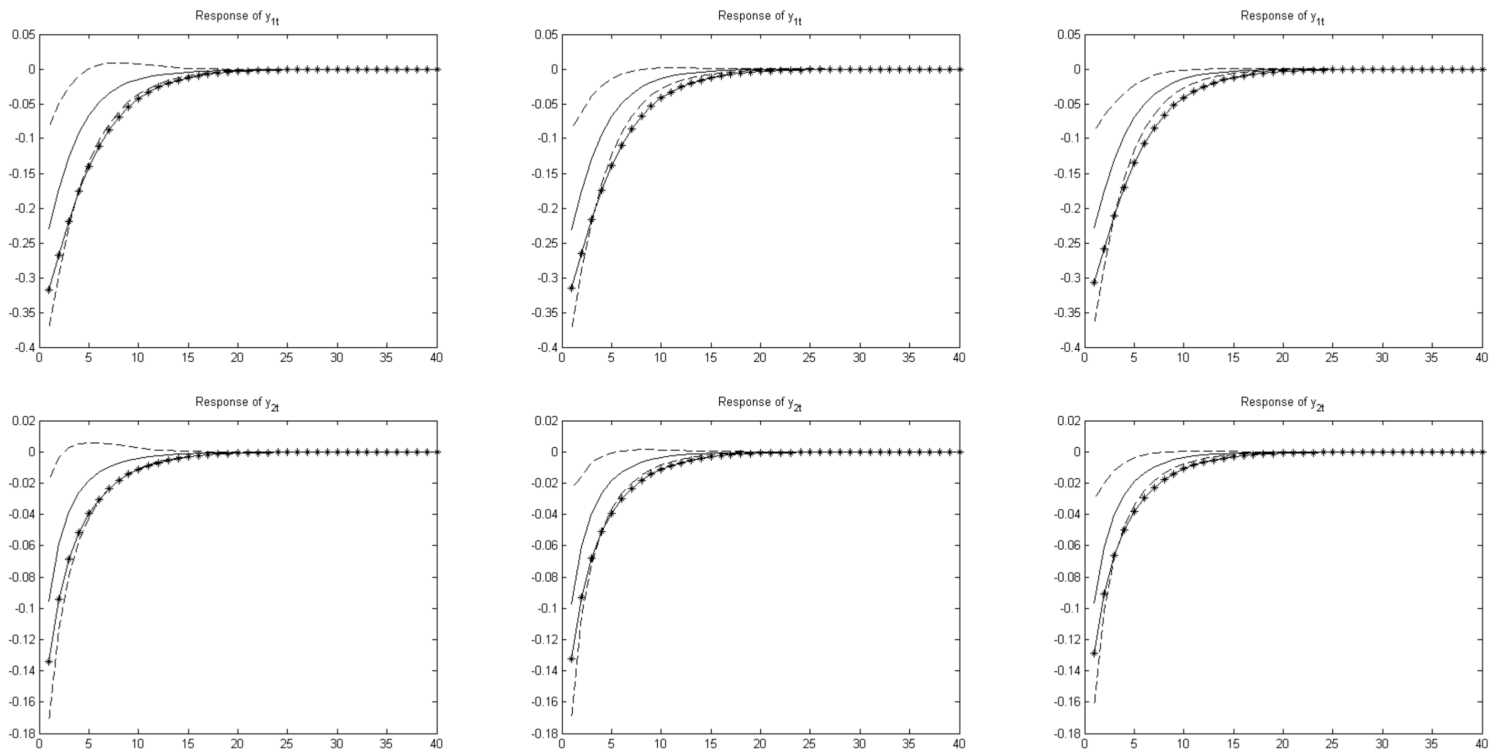
Note: True impulse response functions (–*) based on a random variable bivariate VAR(1) model following the design in Section (6.1) are compared with estimated ones (–*) with two standard-error bands (- -) calculated by empirical interquartile range scaled by the one calculated using normal distribution. The quantile value is set to $\tau = 0.75$. Data is generated using standardized Student's t distribution with 3 degrees of freedom.

Figure 6: Estimated Impulse Responses to a Shock in y_{1t} for $\tau = 0.90$.



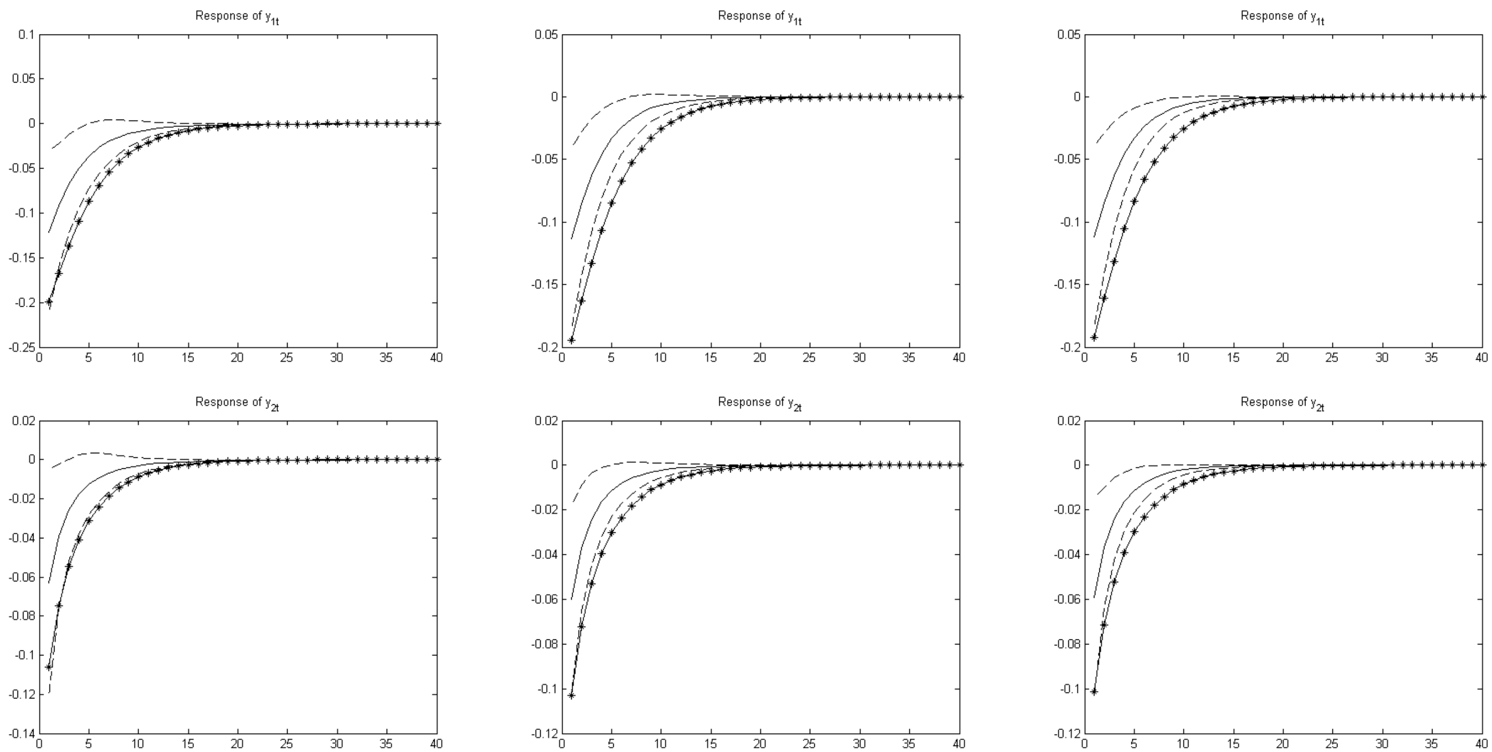
Note: True impulse response functions (–*) based on a random variable bivariate VAR(1) model following the design in Section (6.1) are compared with estimated ones (–*) with two standard-error bands (- -) calculated by empirical interquartile range scaled by the one calculated using normal distribution. The quantile value is set to $\tau = 0.90$. Data is generated using standardized Student's t distribution with 3 degrees of freedom.

Figure 7: Estimated Impulse Responses to a Shock in y_{1t} for $\tau = 0.10$.



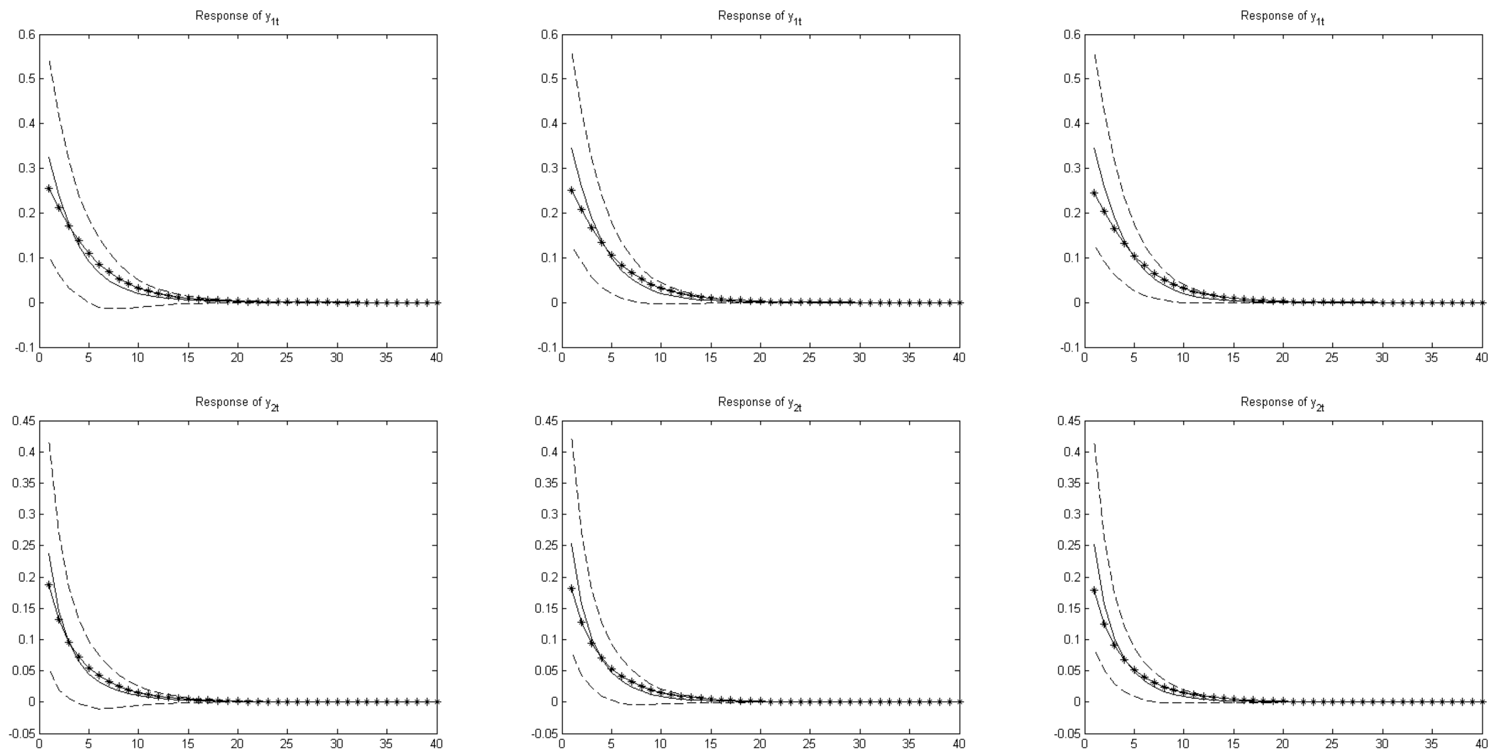
Note: True impulse response functions (–*) based on a random variable bivariate VAR(1) model following the design in Section (6.1) are compared with estimated ones (–*) with two standard-error bands (- -) calculated by empirical interquartile range scaled by the one calculated using normal distribution. The quantile value is set to $\tau = 0.10$. Data is generated using standardized Chi-squared distribution with 3 degrees of freedom.

Figure 8: Estimated Impulse Responses to a Shock in y_{1t} for $\tau = 0.25$.



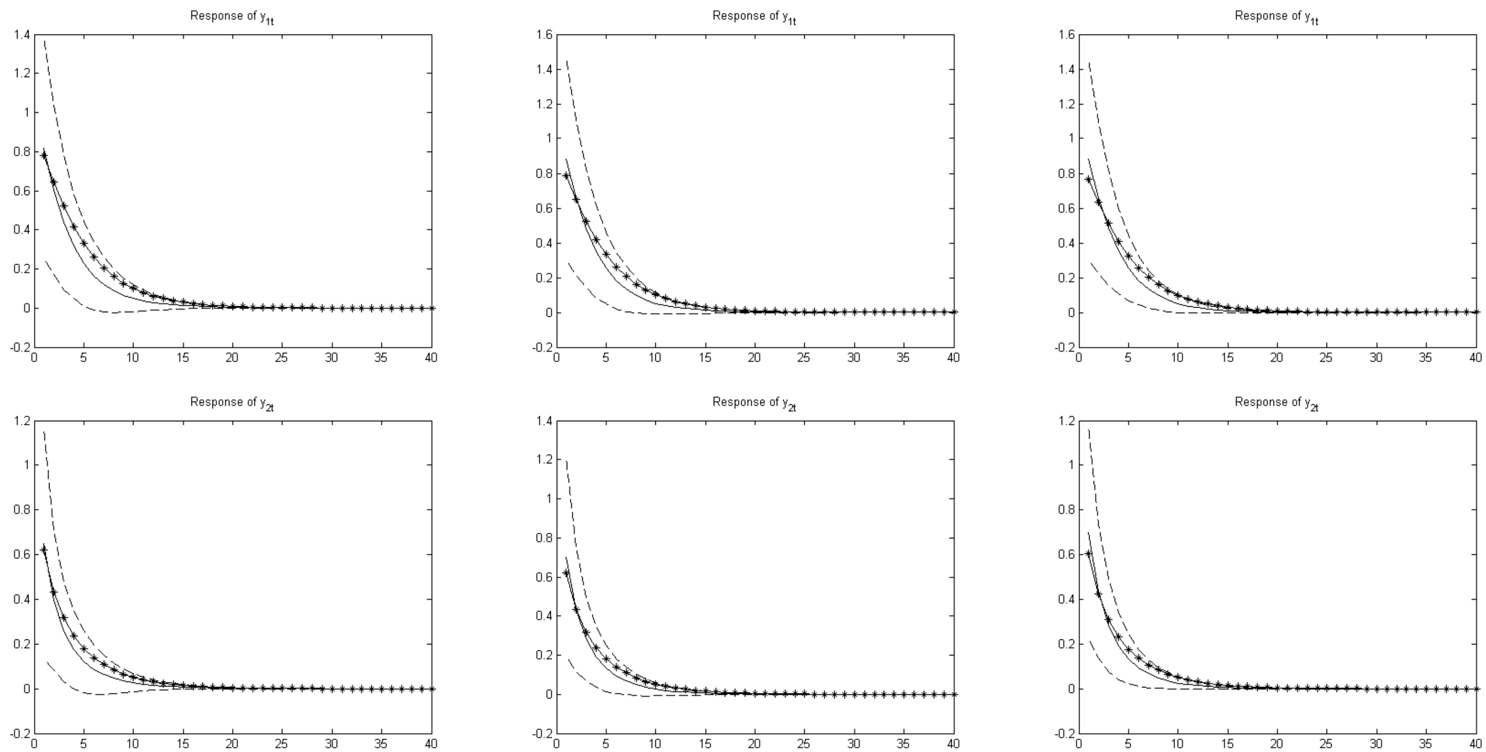
Note: True impulse response functions based on a random variable bivariate VAR(1) model following the design in Section (6.1) are compared with estimated ones with two standard-error bands calculated by empirical interquartile range scaled by the one calculated using normal distribution. The quantile value is set to $\tau = 0.25$. Data is generated using standardized Chi-squared distribution with 3 degrees of freedom.

Figure 9: Estimated Impulse Responses to a Shock in y_{1t} for $\tau = 0.10$.



Note: True impulse response functions based on a random variable bivariate VAR(1) model following the design in Section (6.1) are compared with estimated ones with two standard-error bands calculated by empirical interquartile range scaled by the one calculated using normal distribution. The quantile value is set to $\tau = 0.75$. Data is generated using standardized Chi-squared distribution with 3 degrees of freedom.

Figure 10: Estimated Impulse Responses to a Shock in y_{1t} for $\tau = 0.90$.



Note: True impulse response functions based on a random variable bivariate VAR(1) model following the design in Section (6.1) are compared with estimated ones with two standard-error bands calculated by empirical interquartile range scaled by the one calculated using normal distribution. The quantile value is set to $\tau = 0.90$. Data is generated using standardized Chi-squared distribution with 3 degrees of freedom.

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