A Volatility-Induced Stationary Term Structure Model

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Abstract

I present a novel model of the term structure of interest rates in which mean reversion is induced through level-dependent conditional volatility. The model reconciles unit roots and cointegration with global stationarity and thus captures the extremely persistent but mean-reverting behavior of interest rates. I introduce an approximation that enables analytical computation of model-implied bond yields. In an empirical application with macro-finance risk factors, I show that (i) the short rate exhibits volatility-induced stationarity, which (ii) affects the long-run dynamics; (ii) term premia implied by the volatility-induced stationary model are economically plausible and consistent with survey forecasts; and (iii) the model improves out-of-sample forecasting performance compared to a Gaussian affine term structure model.

Keywords: Yield curve, volatility-induced stationarity, term premia, macro-finance term structure model, unit roots, reduced rank, zero-lower bound.

JEL classification: C32, E43, G12.

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1 Introduction

Interest rates are extremely persistent but globally stationary processes: in short samples indistinguishable from random walks but mean-reverting over longer periods. In addition, the term structure of interest rates appears to be cointegrating as yields of different maturities co-move in the long run. These properties are puzzling in affine term structure models in which unit roots and cointegration are irreconcilable with global stationarity. This paper resolves this puzzle by introducing a novel dynamic term structure model that can simultaneously accommodate unit roots, cointegration, and global stationarity through volatility-induced stationarity.

The paper emphasizes that modeling unit roots and cointegration while maintaining mean reversion is important for the decomposition of the term structure into expectations about future short rates and term premia. In a well-known speech Ben Bernanke (2006) argues that monetary policy makers are highly dependent on models that reliably disentangles these two components: if the current low long-term interest rates reflect skepticism about the future economy, a monetary policy expansion is appropriate. On the other hand, low term premia stimulate financial markets, which calls for a policy tightening. Thus, the proposed model has both theoretical and practical impact.

The key feature of the proposed model is that it exhibits volatility-induced stationarity such that a time-varying conditional volatility drive mean-reversion in the process. Ling (2004) and Nielsen & Rahbek (2014) show that autoregressions with volatility-induced stationarity modeled by a level-dependent conditional volatility can be globally stationary even in presence of unit roots. Intuitively, large realizations of the process are followed by periods in which the stochastic component is weighted by a large conditional variance. Thus, the stochastic term eventually dominates the unit-root conditional mean. This mechanism has also been studied in detail in Albin et al. (2006) and Nicolau (2005). Apart from these non-linear factor dynamics, my model is akin to the discrete-time Gaussian affine term structure model (GATSM).

The Volatility-induced stationary Term Structure Model (VTSM) does not admit a closed-form expression of zero-coupon bond yields. I therefore propose an approximation in which model-implied yields are quadratic in the risk factors. In this sense, the model is related to the class of quadratic term structure models (QTSMs) studied in Ahn et al. (2002), Leippold & Wu (2002), and Realdon (2006). However, whereas the quadratic component in QTSMs comes from a quadratic specification of the short rate, the model studied here generates a quadratic term structure from the conditional variance.

The approximation is sufficiently accurate for the purposes pursued in this paper. In particular, I consider an empirical setting in which the US Treasury bond yield curve is driven by the short and long nominal interest rates, inflation, and a measure of real activity. These processes are extremely persistent and I find evidence on cointegration with two long-term stable relationships. I show that the cointegrated GATSM raises puzzles regarding the adjustment to these long-run relations that are resolved by introducing volatility-induced stationarity. In addition, I find that the VTSM obtains economically plausible term premia estimates that are consistent with evidence provided from survey forecasting data. Finally, the VTSM outperforms the GATSM in out-of-sample forecasting.

Volatility-induced stationarity in interest rate data was first studied by Conley et al. (1997), who consider Markov diffusion models with constant volatility elasticity as in the CKLS model in Chan et al. (1992). Conley et al. (1997) apply these models to overnight effective federal funds rates and conclude that *"when interest rates are high, local mean reversion is small and the mechanism for inducing stationarity is the increased volatility"*. Nicolau (2005) also shows that the federal funds rate can be modelled by a process that exhibits volatility-induced stationarity. Nielsen & Rahbek (2014) extend these analyses by modeling two interest rates, namely the one- and three-month Treasury bill rates, in a bi-variate autoregression with volatility-induced stationarity that allows for cointegration between the rates. This paper contributes to this literature by (i) proposing a no-arbitrage model for the entire term structure that exhibits volatility-induced stationarity and (ii) allowing for more than two variables as driving factors of the term structure.

In a broader context, the paper provides insights in the discussion of the persistence problem, i.e., that interest rates are more persistent than captured by stationary vector autoregressions, and its implications for estimation of term premia. Shiller (1979) notes that models that revert faster to a mean level than the data attribute almost all variation in long-term interest rates to term premia. This is counterfactual according to survey data, which suggest that term premia are stable an mainly driven by expectations (Kim & Orphanides, 2007, 2012). On the other hand, non-stationary models do not allow for any mean-reversion in interest rates. In result, expected future short rates are close to current rates and term premia are close to constant. However, Campbell & Shiller (1991) and Backus et al. (2001) find evidence of time-varying premia in yields and forward rates. The VTSM provides a solution to this puzzle by allowing for unit roots in a stationary model.

Other methodologies have been suggested to overcome the persistence problem. One strand of literature focuses on the well-known statistical problem that the autoregressive parameter of stationary VAR models is downwardly biased in small samples when data is persistent. To tackle this problem, Kim & Orphanides (2007, 2012) augment the data with survey forecasts and Bauer et al. (2014) suggest a bias-correction that results in stable term premia. This approach is conceptually different from that taken in this paper in which linear dynamics is abandoned to introduce nonlinearity in the form of volatility-induced stationarity. Abbritti et al. (2016) and Goliński & Zaffaroni (2016) suggest that long memory represents a realistic, intermediate case between stationary I(0) and non-stationary I(1) extremes of affine term structure models. Further, to capture the long-run co-movement between yields of different maturities, Osterrieder (2013) considers fractional cointegration and Jardet et al. (2013) study a model with near-cointegration modeled by combining stationary and unit root models with model averaging techniques.

The paper is also related to stochastic volatility affine term structure models that are studied and classified in Dai & Singleton (2000). As in the VTSM, these models specify level-dependent conditional volatilities. There are, however, important differences. Firstly, whereas one must impose adverse parameter restrictions to ensure positive definite covariance matrices in affine models, the conditional variance specification in the VTSM is positive definite by construction. Secondly, in contrast to the VTSM, affine models constrain factor processes that drive conditional heteroskedasticity to the positive domain. This side-effect is particular problematic in macro-finance models as the one considered in this paper since macro risks such as inflation and growth rates can become negative.

Section 2 presents the volatility-induced stationary term structure model and approximate computation of model-implied bond yields. The section includes a detailed comparison between the model and respectively the affine and quadratic classes of term structure models. In Section 3, I analyze the data, construct a set of risk factors, and estimate a vector autoregression with reduced rank and volatility-induced stationarity. Given these estimated factor dynamics, Section 4 considers the term structure implications of volatility-induced stationarity and analyze model-implied term premia. Section 5 concludes.

2 Volatility-Induced Stationary Term Structure Model

2.1 Factor Dynamics

I set up a model in which the term structure is driven by both yield and macro risks. The yield risks are summarized by the short rate, r_t, and a long-maturity yield, R_t. I model macro risks by an inflation measure, π_t, and a measure of real activity, g_t. Let X_t denote a state vector containing these risks: $X_t = [r_t,\ R_t,\ \pi_t,\ g_t]'$.

The primary challenge of this paper is to model factor dynamics that are highly persistent but remain globally stationary. For this purpose, I propose a factor process that can induce stationarity through the conditional factor volatilities. In particular, consider the double-autoregressive process (Ling, 2004, Nielsen & Rahbek, 2014); a discrete-time, autoregressive model with level-dependent conditional covariance matrix:

$$\begin{aligned} X_{t+1} &= \mu + \Phi X_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} = \Omega_t^{1/2}\epsilon_{t+1} \\ \epsilon_{t+1} &\sim \text{ i.i.d. } \mathcal{N}\left(\mathbf{0}, I_4\right) \\ \Omega_t &= \Omega + \Gamma X_t X_t' \Gamma', \end{aligned} \tag{1}$$

where Ω is symmetric positive definite matrix, which ensures that Ω_t is positive definite.

I leave Φ, Ω, and Γ fully parameterized allowing for dependence between all risk factors. This contrasts other studies in the macro-finance term structure literature in which yield and macro risks are assumed to be orthogonal (Ang & Piazzesi, 2003, Ang et al., 2006, Goliński & Zaffaroni, 2016, Monfort & Pegoraro, 2007).

Volatility-induced stationarity

The process in (1) is stationary and geometrically ergodic if its top Lyapunov exponent is strictly negative (Nielsen & Rahbek, 2014, Theorem 1). Importantly, this restriction does not exclude cases with unit roots in the characteristic polynomial. To understand the intuition behind this result, consider a univariate example with $\Phi = 1$. If $\Gamma = 0$, the process is obviously a random walk and non-stationary. With $\Gamma \neq 0$, however, a large value of X_t results in a large conditional variance of X_{t+1}. Thus, extended periods of increasing realizations eventually imply that the stochastic component dominates the unit-root conditional mean. In result, when a negative innovation is realized to a process that is far above its mean, the process is pushed towards its mean level. In this sense, the conditional volatility can induce mean-reversion in a unit root vector autoregression.

Cointegration and Short-Run Structure

I allow for reduced rank to model cointegration between the factors. Assuming that the characteristic polynomial has q unit roots and $(4-q)$ roots outside the unit circle, the rank of $(\Phi - I_4)$ in (1) is $r = 4 - q$. Then, Φ can be parameterized by $\Phi = I_4 + \alpha\beta'$, where α and β are $4 \times r$ matrices. Thus, the model can be written in error-correction form by

$$\Delta X_t = \mu + \alpha\beta' X_{t-1} + \sum_{k=1}^{K} \Psi_k \Delta X_{t-k} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \Omega_{t-1}) \tag{2}$$

To achieve a well-specified model, I will let the data dictate the lag structure of the model. Note that the error correction model can be re-written in the form of (1). Thus, I will abstract from the error-correction model in (2) in the further discussion of the model.

2.2 Stochastic Discount Factor

The one-period stochastic discount factor is given by

$$\mathcal{M}_{t+1} = \exp\left(-r_t - \frac{1}{2}\Lambda_t' \Omega_t \Lambda_t - \Lambda_t' \varepsilon_{t+1}^{\mathbb{Q}}\right), \tag{3}$$

where $\varepsilon_{t+1}^{\mathbb{Q}} = \Omega_t^{1/2} \epsilon_{t+1}^{\mathbb{Q}}$, $\epsilon_{t+1}^{\mathbb{Q}} \sim$ i.i.d. $\mathcal{N}(0, I_4)$ and Λ_t is the market price of risk measured by conditional variance, Ω_t. I specify the market price of risk such that the factor dynamics also follow a DAR process under the risk-neutral $\mathbb{Q}$-measure. Thus, let

$$\Lambda_t = \Omega_t^{-1}(\lambda_0 + \lambda_1 X_t) \tag{4}$$

which result in $\mathbb{Q}$-dynamics given by

$$X_{t+1} = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_t + \varepsilon_{t+1}^{\mathbb{Q}} \tag{5}$$

with $\mu^{\mathbb{Q}} = \mu - \lambda_0$, $\Phi^{\mathbb{Q}} = \Phi - \lambda_1$, and Ω_t defined in (1). Finally, note that per construction of the state vector, there is the following relationship between the short rate and X_t:

$$r_t = \iota_1' X_t \tag{6}$$

where ι_1' denotes a unit vector with one in its first element. Equations (1)-(6) establish the volatility-induced stationary term structure model, which I coin the VTSM.

2.3 Zero-Coupon Bond Yields

Let $Y_{t,n}$ denote the yield of a bond with price $P_{t,n}$, hence $Y_{t,n} = -\log(P_{t,n})/n$. The no-arbitrage price of a zero-coupon bond expiring $(n+1)$ periods from the current time t is given by

$$P_{t,n+1} = \mathbb{E}_t\left(\mathcal{M}_{t+1}P_{t+1,n}\right).$$

The VTSM does not admit a closed-form bond price expression that satisfies this equation. Instead, I propose the following approximative bond price solution, $\tilde{P}_{t,n}$:

$$\tilde{P}_{t,n} = \exp\left(A_n + B_n'X_t + C_n'\text{vec}\left(X_tX_t'\right)\right), \tag{7}$$

where

$$\begin{aligned}
A_n &= A_{n-1} + B_{n-1}'\mu^{\mathbb{Q}} + C_{n-1}'\left(\text{vec}\left(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}'}\right) + \text{vec}\left(\Omega\right)\right) + \frac{1}{2}B_{n-1}'\Omega B_{n-1} \\
B_n' &= -\iota_1 + B_{n-1}'\Phi^{\mathbb{Q}} + C_{n-1}'\left(\Phi^{\mathbb{Q}}\otimes\mu^{\mathbb{Q}} + \mu^{\mathbb{Q}}\otimes\Phi^{\mathbb{Q}}\right) \\
C_n' &= C_{n-1}'\left(\Phi^{\mathbb{Q}}\otimes\Phi^{\mathbb{Q}} + \Gamma\otimes\Gamma\right) + \frac{1}{2}\left(\left[B_{n-1}'\Gamma\right]\otimes\left[B_{n-1}'\Gamma\right]\right)
\end{aligned}$$

initiated at $n = 0$ with $A_0 = 0$, $B_0 = \mathbf{0}_{p\times 1}$, $C_0 = \mathbf{0}_{p^2\times 1}$.

2.4 Enforcing the Zero-Lower Bound

The bond price expression in (7) allow us to formulate restrictions on the loading recursions that ensure positive yields across the entire term structure.

Lemma 1 *The VTSM enforces the zero-lower bound for yields with $n \geq 2$, if the loading recursions satisfy $\frac{1}{4}Bn'\tilde{C}_n^{-1}B_n - A_n \geq 0$ and $\tilde{C}_n$, defined such that vec$\left(\tilde{C}_n\right) = C_n$, is positive definite.*

Proof. The quadratic term in (7) can be re-written in the following way

$$C_n'\text{vec}\left(X_tX_t'\right) = \left(X_t\otimes X_t\right)'\,\text{vec}\left(\tilde{C}_n\right) = X_t'\tilde{C}_nX_t.$$

Assuming $\tilde{C}_n$ is positive definite, the model-implied bond yield takes it minimum at

$$-n\frac{\partial Y_{t,n}}{\partial X_t} = B_n + 2\tilde{C}_nX_t = 0 \quad\Leftrightarrow\quad X_t = -\frac{1}{2}\tilde{C}_n^{-1}B_n.$$

Thus, the lower bound is given by

$$Y_{t,n} \geq -\frac{1}{n}\left(A_n - \frac{1}{4}B_n'\tilde{C}_n^{-1}B_n\right)$$

which satisfies the zero-lower bound if $\frac{1}{4}Bn'\tilde{C}_n^{-1}B_n - A_n \geq 0$. $\square$

2.5 The VTSM versus Affine and Quadratic Term Structure Models

Affine term structure models

Time-varying conditional variances can be modeled in the affine framework by specifying the factor dynamics as a mixture of Gaussian and gamma autoregressive processes. Dai & Singleton (2000) introduce the notation $\mathcal{A}_m(p)$, for affine models with p factors, where $m \leq p$ factors control the stochastic volatilities of all factors. Thus, the GATSM is denoted by $\mathcal{A}_0(p)$.

The VTSM generally allows all factors to affect the conditional variance matrix akin to $\mathcal{A}_p(p)$ models but with non-linear dynamics. The non-linearity in the VTSM brings two advantages over the $\mathcal{A}_p(p)$ class. First, the conditional state variance matrix, Ω_t, is positive definite so long as the constant component, Ω, is positive definite. This restriction is straightforwardly implemented by a Cholesky decomposition with strictly positive diagonal elements. In contrast, the parameter restrictions necessary for ensuring admissibility of affine models limit the flexibility in modeling conditional correlations to an extend that affects the empirical performance of these models. At the extreme, Dai & Singleton (2000) show that the class of $\mathcal{A}_p(p)$ models require that state variables are conditionally uncorrelated and that unconditional correlations are non-negative.

Second, VTSMs do not constrain the state variables to the positive domain, whereas gamma autoregressive processes are positive by construction. Obviously, when the state vector contains macroeconomic variables such as growth in real activity or inflation rates, such constraints are disadvantageous. In the $\mathcal{A}_m(p)$ class, $(p - m)$ factors are Gaussian and can in fact become negative. But these same factors cannot control the conditional volatility. In practice, this limitation implies that affine term structure models are not suitable for applications in which macro risks drive time-varying volatility of the yield curve. In the empirical part of this paper, I do indeed find that inflation and real activity drives the conditional volatility matrix of the state vector.

Quadratic term structure models

Due to the quadratic form of (7), it is natural to compare the VTSM with the class of quadratic models (QTSMs) studied in Leippold & Wu (2002), Ahn et al. (2002), and Realdon (2006). QTSMs assume a quadratic short rate of the form,

$$r_t = \gamma_0 + \gamma_1' X_t + X_t' \Psi X_t.$$

When the factors follow vector autoregressive dynamics and the stochastic discount factor is specified as in (3), the model has a closed-form bond price solution given by

$$\check{P}_{t,n} = \exp\left(\check{A}_n + \check{B}_n' X_t + X_t' \check{C}_n X_t\right)$$

with loadings given recursively in Realdon (2006). This expression can be written in the same form as (7) (see the proof of Lemma 1). It follows that the QTSM and VTSM can produce similar shapes of the yield curve abstracting from the no-arbitrage restrictions implicit in the recursions. However, the source of the quadratic term and thus the loading recursions are highly different across the two model frameworks: whereas the quadratic bond yield stems naturally from the variance specification in the VTSM, the QTSM imposes this non-linearity through an arbitrary specification of the short rate. Thus, the VTSM is conceptually different from the class of QTSMs.

This difference is particularly highlighted in the macro-finance model considered in this paper in which the short rate is a factor itself. In this setting, the short-rate specification is linear per construction and the QTSM reduces in this case to the Gaussian ATSM.

3 Empirical Analysis of Risk Factors

3.1 The Data

The yield data is US Treasury zero-coupon bond yields sampled monthly (end-of-month) between January 1985 and December 2016 with maturities $1, 2, ..., 10$ years from Gürkaynak et al. (2007). The sample has 384 observations in the time-series dimension. I augment this data with the one-month Treasury Bill rate.[1] Thus, the short and the long rates are given by respectively the one-month and ten-year yields, $r_t = Y_{t,1}$ and $R_t = Y_{t,12}$.

The macro risk factors are constructed following the approaches in Ang & Piazzesi (2003) and Goliński & Zaffaroni (2016). The inflation measure is thus given by the first principal component of standardized series of CPI and PPI data from the US Bureau of Labor Statistics. Analogously, The measure of real activity is the first principal component of standardized data on the unemployment and employment growth rates from the US Bureau of Labor Statistics; the industrial production index from Federal Reserve

[1]This is extracted from the Fama/French factor files available at the authors' website, `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`.

Economic Data; and the help-wanted-advertising-in-newspapers (HELP) index from Barnichon (2010).

Table 1 characterizes these macro risks in terms of the observed data series. The inflation factor is equally correlated with each of the inflation data series, while the factor on real activity correlates strongest with employment growth rate and the HELP index. Correlation with the unemployment rate is negative as expected. In total, the inflation and real activity factors account for respectively 85 and 66 pct of the total variation in the underlying data series.

The risk factors are exhibited in Figure 1. We observe that the processes are extremely persistent but fluctuate around stationary levels throughout the sample. Particularly the short rate appears to exhibit volatility-induced stationarity: the process is clearly heteroskedastic with higher volatility in periods with higher short-rate levels. Towards the end of the sample, where the short rate is absorbed at the zero-lower bound, the short-rate is very stable. In addition, the process exhibits occasional jumps.

3.2 Unit Roots and Reduced Rank

The risk factors are cross-correlated with coefficients reported in Table 2. Hence, the standard assumption of orthogonality between yield and macro risks is inappropriate and a full parametrization of the autoregressive parameter matrix Φ and the parameters of the conditional covariance matrix appears to be necessary. The table also report autocorrelations supporting a local unit root behavior of the factors. Formal unit root and stationarity tests detect unit roots in the yield factors but not in the macro factors, as shown in Table 9 in appendix.

I proceed by testing for cointegration by the Johansen (1991) test based on the linear

	Explained (pct)	Correlations					
		CPI	PPI	UNEMP	EMP	PROD	HELP
π_t	85.49	0.92	0.92	-	-	-	-
g_t	65.75	-	-	-0.71	0.93	0.71	0.87

Table 1: The table reports the percentage of variation in respectively inflation data (CPI, PPI) and data related to real activity (UNEMP, EMP, PROD, HELP) explained by respectively the inflation measure (π_t) and the real activity measure (g_t). Correlations between these measures and the underlying variables is shown as well.

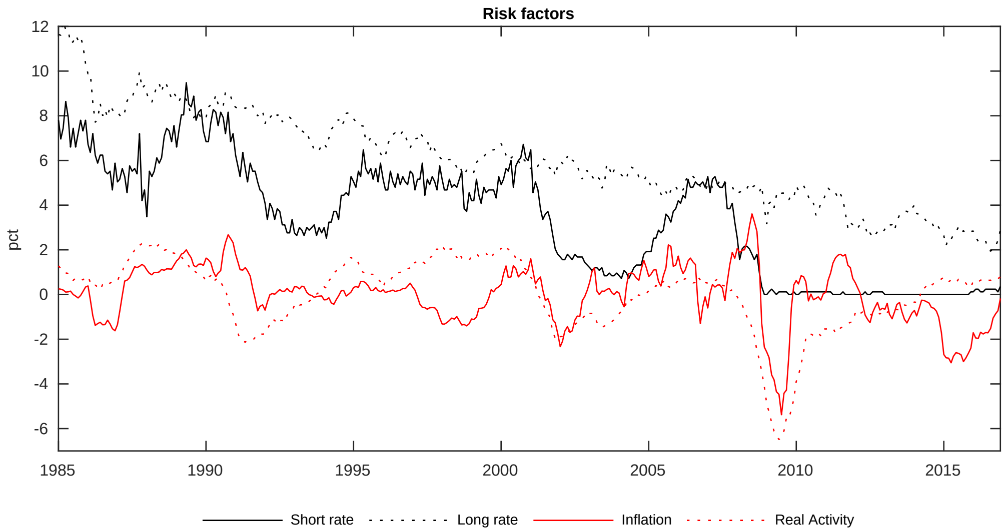

Figure 1: Risk factors.

	r_t	R_t	g_t	π_t	r_{t-1}	R_{t-1}	g_{t-1}	π_{t-1}
r_t	1.00	0.87	0.42	0.56	0.98	0.87	0.42	0.58
R_t	0.87	1.00	0.39	0.37	0.86	0.99	0.39	0.37
π_t	0.42	0.39	1.00	0.29	0.42	0.38	0.96	0.30
g_t	0.56	0.37	0.29	1.00	0.54	0.36	0.27	0.99

Table 2: Auto- and cross-correlations in the risk factors

vector error correction model, i.e., (2) with $\Gamma = \mathbf{0}_{p\times p}$. The lag structure in the short-run dynamics in this model is determined by general-to-specific LR tests, information criteria, and misspecification tests. For the choice of three-months lags, the residuals are not autocorrelated according to univariate Ljung-Box tests. The Johansen test, for which results are reported in Table 10 in appendix, suggests a reduced rank of $r = 2$, hence, two common stochastic factors drive the yield curve and there are two long-run stable relations between the risk factors.

3.3 Factor Dynamics Estimation

The VTSM factor dynamics are thus given by the error-correction model in (2) with reduced rank $r = 2$ and lag length $K = 3$. I estimate this model by maximum likelihood[2]

[2]Nielsen & Rahbek (2014) prove that maximum likelihood estimators in a particular bi-variate specification of the model in (1) are consistent and asymptotically normal. Since these results do not generalize to the specification considered here, I confirm by simulations that the maximum likelihood estimator

under just-identifying restrictions. Then, I impose further restrictions to obtain models with economically sensible interpretations, which I comment on below.[3] I repeat this procedure for the GATSM by setting $\Gamma = \mathbf{0}_{p\times p}$. Since the reduced-rank GATSM is identical to the VTSM in absence of volatility-induced stationarity, I use it as reference throughout the empirical application.

Tables 3 provides the estimated long-run dynamics and short-run estimates are placed in Appendix A, Table 11. The VTSM obtains the lowest value of the AIC and the likelihood values of the models are significantly different when compared by a LR test. Likewise, misspecification tests reported in Table 12 in appendix suggest that the time-varying conditional covariance matrix of the VTSM removes autocorrelation and improves normality tests of the standardized residuals in comparison with the linear dynamics in the GATSM.

We also note that the estimated top Lyapunov exponent in the VTSM is strictly negative.[4] Therefore, the volatility-induced stationary factor process is stationary, hence compatible with mean-reverting yield processes despite unit roots in the yield factors. This feature contrasts the GATSM dynamics in which unit roots imply global non-stationarity. We shall see below that this difference carry implications for the long-run adjustments and term premia implied by the models.

3.4 Sources of Volatility-Induced Stationarity

The VTSM allows all factors to exhibit volatility-induced stationarity and furthermore, the conditional heteroskedasticity can be driven by all factors. This general setting allows me to make statements about the estimated sources of volatility-induced stationarity. Table 4 report standard deviations of all conditional covariances between the risk factors and how each of these correlate with the risk factors.

The short rate stands out with a highly volatile conditional variance, which is driven mainly by the yield curve factors. The macro factors also correlate with the conditional short-rate variance, but I note that the estimated coefficient on the real activity measure is insignificant (see Table 11, Appendix A). Also the conditional covariances between the

exhibit reasonable properties for the four-factor DAR model. The simulations are available upon request.

[3]The restrictions are imposed sequentially starting with setting the most insignificant estimates to zero first in the cointegrating vector and then in the adjustment matrix. At each step, the restrictions are tested by LR tests and the short-run coefficient estimates are compared.

[4]The Lyapunov exponents are obtained by the efficient and numerically stable algorithm described in Nielsen & Rahbek (2014). Standard errors are extremely computationally demanding in this system of 16 exponents and therefore, not reported.

Long-run factor dynamics in the VTSM:

$$\Delta \begin{pmatrix} r_t \\ R_t \\ \pi_t \\ g_t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \underset{(0.012)}{-0.031} \\ \underset{(0.008)}{0.017} & 0 \\ \underset{(0.003)}{0.008} & 0 \end{pmatrix} \nu_{t-1} + \textstyle\sum_{i=1}^{3} \hat{\Gamma}_i \Delta X_{t-i} + \hat{\mu} + \varepsilon_t,$$

$$r_t = \underset{(1.150)}{3.672}\ \pi_t + \underset{(0.611)}{1.681}\ g_t + \nu_{t,1},$$

$$R_t - r_t = \nu_{t,2}.$$

Log-likelihood: 6986, AIC$= -13806$, $\dim(\theta_X) = 83$, $T = 384$

Top-Lyapunov Exponent: -0.004.

Long-run factor dynamics in the GATSM:

$$\Delta \begin{pmatrix} r_t \\ R_t \\ \pi_t \\ g_t \end{pmatrix} = \begin{pmatrix} \underset{(0.008)}{-0.016} & \underset{(0.027)}{0.050} \\ 0 & 0 \\ \underset{(0.009)}{0.017} & \underset{(0.017)}{-0.033} \\ \underset{(0.003)}{0.009} & 0 \end{pmatrix} \nu_{t-1} + \textstyle\sum_{i=1}^{3} \hat{\Gamma}_i \Delta X_{t-i} + \hat{\mu} + \varepsilon_t,$$

$$r_t = \underset{(0.911)}{3.097}\pi_t + \underset{(0.715)}{2.174} g_t + \nu_{t,1},$$

$$R_t - r_t = \nu_{t,2}.$$

Log-likelihood: 6823, AIC: -13507, $\dim(\theta_X) = 69$, $T = 384$.

Table 3: Estimates of cointegrating relations and adjustment speeds in the factor dynamics of the VTSM and GATSM. Estimated top Lyapunov exponent is reported for the former. Standard errors in parentheses. Remaining estimation results are reported in Table 11 in Appendix A.

	std	corr(·,r_t)	corr(·,R_t)	corr(·,π_t)	corr(·,g_t)
$\text{var}_{t-1}(r_t)$	**2.32**	0.93	0.82	0.40	0.48
$\text{var}_{t-1}(R_t)$	**0.04**	-0.07	-0.04	-0.22	-0.43
$\text{var}_{t-1}(\pi_t)$	**0.29**	-0.31	-0.12	-0.34	-0.84
$\text{var}_{t-1}(g_t)$	0.01	0.03	0.05	-0.29	-0.42
$\text{cov}_{t-1}(r_t, R_t)$	**0.24**	0.75	0.50	0.14	0.68
$\text{cov}_{t-1}(r_t, \pi_t)$	**0.36**	0.17	0.32	0.25	-0.40
$\text{cov}_{t-1}(r_t, g_t)$	0.08	0.80	0.62	0.50	0.56
$\text{cov}_{t-1}(R_t, \pi_t)$	**0.11**	0.42	0.25	0.35	0.77
$\text{cov}_{t-1}(R_t, g_t)$	0.02	0.07	0.09	-0.31	-0.39
$\text{cov}_{t-1}(\pi_t, g_t)$	0.05	0.30	0.18	0.53	0.64

Table 4: Standard deviations and correlations with risk factors of the elements of the estimated conditional covariance matrix, $\hat{\Omega}_{t-1}$. Boldfaced standard deviations for covariances that are significantly time-varying (associated $\hat{\Gamma}_{ij}$ significant on the 5% level for at least one i, j.).

short rate, long rate, and inflation measure along with the conditional variance of inflation exhibit volatility-induced stationarity although to a lesser extent than that of the short rate. Thus, the short rate is the primary source of volatility-induced stationarity in the term structure. The variance of the real activity measure and its associated covariances are very stable and the corresponding estimated coefficients of Γ are insignificant. To this end we recall that the unit root test was rejected for this variable.

The estimated conditional covariances that involve the short rate are exhibited in Figure 2. As expected, the variances implied by the VTSM are fluctuating around the constant levels implied by the GATSM. I also observe that the conditional covariances are highly time-varying and only stay near zero during the zero-lower bound regime. These patterns cannot be replicated by the affine $\mathcal{A}_p(p)$ models in Dai & Singleton (2000) due to the admissibility restrictions discussed earlier.

3.5 Long-Run Dynamics and Stationarity

The VTSM and the GATSM carry similar predictions concerning the long-run stable relations: the long and short rates are cointegrated as in Hall et al. (1992) and there is a long-run co-movement between the short rate, inflation, and real activity. Since the short rate follows the Federal Funds rate closely, this relation mimics the dual mandate of the

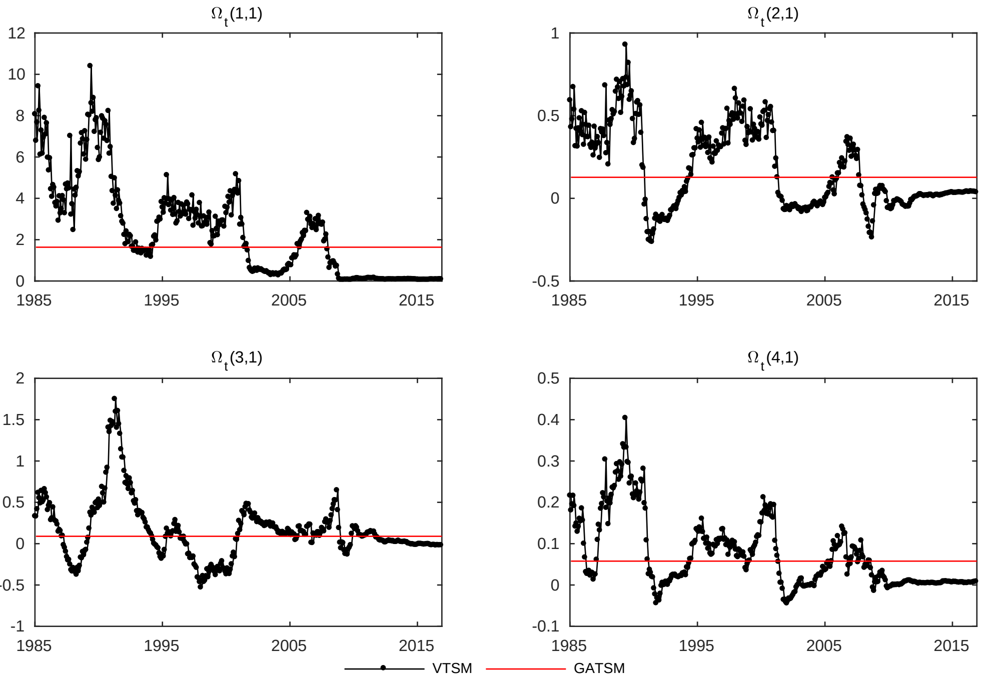

Figure 2: Estimated conditional variance of the short rate and conditional covariance between the short rate and the remaining factors; $\Omega_t(i,1)$ denotes the covariance with the i'th risk factor in the state vector. Reported in basis points.

Federal Reserve (Fed). In addition, the signs of the estimates are intuitive: a low interest rate is associated with high levels of inflation and real activity.

The models, however, produce different estimates of the adjustment matrix. In the VTSM, the long-run stability of the yield spread is maintained by the long rate, whereas the GATSM predicts that the short rate adjust towards this equilibrium. To the extend that the yield spread equilibrium reflects no-arbitrage, we should expect the long rate, i.e., market mechanisms, to erode short-lived arbitrage opportunities rather than the short rate, which is primarily controlled by the Fed.[5] As for the second equilibrium, the models disagree on whether the short rate enters the error-correcting mechanism jointly with the macro factors. The fact that the Fed determines the policy discretionally rather than mechanically from macro risks (Bernanke, 2015) is an argument for the plausibility of the VTSM, in which the short rate is not significant in the adjustment matrix.

The common stochastic factors of the yield curve are determined by the orthogonal complement of the adjustment matrix, $\alpha_\perp$, as $\hat{\alpha}'_\perp \sum_{s=1}^{t} \hat{\varepsilon}_s$, where $\hat{\varepsilon}$ denotes the standardized residual. These are plotted in Figure 3. By comparing with the risk factor plot in

[5]While the FOMC does use information on developments in the long-term yield to make decisions about the discount rate (Bernanke, 2006), it is not concerned with maintaining a stable spread.

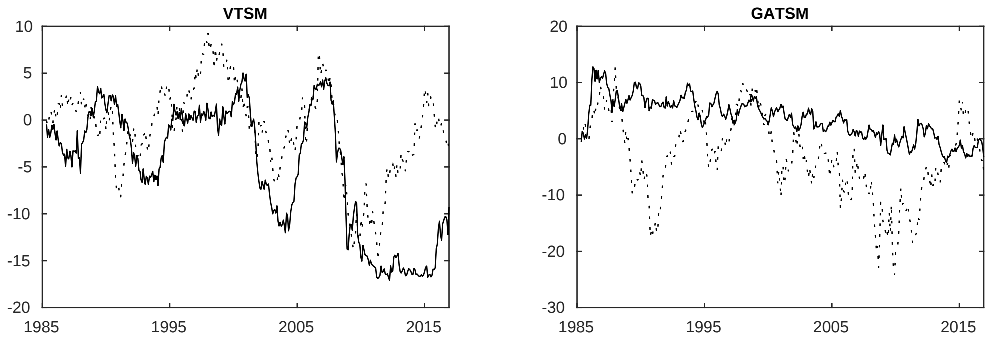

Figure 3: Common stochastic factors computed by $\hat{\alpha}_{\perp}' \sum_{s=1}^{t} \hat{\varepsilon}_s$ for the VTSM and GATSM.

Figure 1, we identify that the VTSM is partly driven by shocks to the short rate. Hence, the VTSM predicts that the Fed can affect the dynamics of the entire yield curve by controlling the short rate and that the market will adjust to such monetary policy shocks.

The GATSM, on the other hand, is partly driven by shocks to the long rate. A similar conclusion is reached by Hall et al. (1992) also in an affine model but with yields only. They interpret these results as evidence of the expectations hypothesis: the term structure is driven by the long rate to which current short rates adjust using the information contained in the yield spread. We shall see in Section 4.2 that the GATSM is indeed consistent with the expectations hypothesis as it implies almost constant term premia.

In conclusion, I find that the short rate is an important factor for establishing mean reversion in the factor dynamics. In the GATSM, this property is realized as an error-correcting behavior that is economically implausible. In contrast, the VTSM allows the short rate to induce stationarity through the conditional volatility.

4 Empirical Term Structure Results

4.1 Term Structure Estimation

The error-correction model analyzed thus far can be re-written as a process that follows the single-lag model in (1) given by $\tilde{X}_t = [X_t', \ X_{t-1}', \ X_{t-2}', \ X_{t-3}']'$ with constant $\tilde{\mu} = [\mu', \mathbf{0}_{1\times 12}]'$ and autoregressive parameter

$$\tilde{\Phi} = \left(\begin{array}{cccc} \alpha\beta' + \mathbf{I}_4 + \Psi_1 & \Psi_2 - \Psi_1 & \Psi_3 - \Psi_2 & -\Psi_3 \\ \hdashline \multicolumn{3}{c}{\mathbf{I}_{12}} & \mathbf{0}_{4\times 4} \end{array} \right).$$

The corresponding dynamics under the $\mathbb{Q}$-measure given by (5) has conditional mean parameters, $\tilde{\mu}^{\mathbb{Q}}$ and $\tilde{\Phi}^{\mathbb{Q}}$. I treat lagged states as unspanned factors. Thus, lagged risk factors are included in the model to forecast current states, but do no explain the term structure cross-section [6]. It follows that

$$\tilde{\Phi}^{\mathbb{Q}} = \begin{pmatrix} \Phi^{\mathbb{Q}} & \Psi_2 - \Psi_1 & \Psi_3 - \Psi_2 & -\Psi_3 \\ \multicolumn{3}{c}{\mathbf{I}_{12}} & \mathbf{0}_{4\times 4} \end{pmatrix}.$$

Thus, $\mu^{\mathbb{Q}}$ and $\Phi^{\mathbb{Q}}$ are free parameters with dimensions of respectively 4×1 and 4×4. I estimate the parameters taking the parameters obtained from the factor dynamics, $\hat{\Theta}^{\mathbb{P}}$, as given.[7] The parameters are fitted to the yield curve given by maturities 1 to 10 years from Gürkaynak et al. (2007). Thus, there is $N = 10$ observations in the cross-section for $T = 384$ months. The estimation problem is given by

$$\min_{\mu^{\mathbb{Q}}, \Phi^{\mathbb{Q}}} \quad \sum_{t=1}^{T} \sum_{i=1}^{N} \left\{ Y_{t,12i} - \hat{Y}_{t,12i} \left(\mu^{\mathbb{Q}}, \Phi^{\mathbb{Q}}, \hat{\Theta}^{\mathbb{P}} \right) \right\}^2$$

where $\hat{Y}_{t,n} \left(\mu^{\mathbb{Q}}, \Phi^{\mathbb{Q}}, \Theta^{\mathbb{P}} \right) = -n^{-1} \log \hat{P}_{t,n} \left(\mu^{\mathbb{Q}}, \Phi^{\mathbb{Q}}, \Theta^{\mathbb{P}} \right)$ with $\hat{P}_{t,n}$ computed in closed form by (7). To achieve convergence of this highly non-linear problem, I first estimate $\Phi^{\mathbb{Q}}$ with $\mu^{\mathbb{Q}} = \mathbf{0}_{p\times 1}$ and then, $\mu^{\mathbb{Q}}$ given the estimated value of $\Phi^{\mathbb{Q}}$. Using these estimates as starting values, the parameters are re-estimated jointly. The results are shown in Table 5.

The persistence of the models under the $\mathbb{P}$-measure is given by maximum eigenvalues equal to one by construction. Under the $\mathbb{Q}$-measure, the VTSM contains an eigenvalue outside the unit circle, equal to -1.025, while the maximum eigenvalue of the GATSM is 0.9999. However, due to the time-varying conditional volatility, the VTSM remains stationary with a top-Lyapunov exponent of -0.017.

4.2 Term Premia

The volatility-induced stationary property of the VTSM affects its implied term premia estimates radically. Figure 4 presents the five-by-five-year forward rate and associated term premia[8] of the VTSM against the GATSM. I compare these term premia to those generated by a stationary Gaussian affine term structure model, in which reduced-rank restrictions are not imposed, computed by Wright (2011). Bauer, Rudebusch, and Wu

[6]Models with unspanned factors are studied in Duffee (2011), Joslin et al. (2014), Ludvigson & Ng (2009), Wright (2011)

[7]This two-step estimation method is a common approach in the macro-finance term structure literature, see for instance Ang & Piazzesi (2003), Ang et al. (2006), Goliński & Zaffaroni (2016).

[8]Computed by the difference between the forward rate and the model-implied forward rate when the market price of risk is zero.

	VTSM				GATSM			
$\mu^{\mathbb{Q}\prime}(\times 10^{-3})$	-2.591	-2.765	1.459	5.371	-0.140	0.066	-0.284	-0.052
	(1.336)	(1.212)	(0.731)	(2.490)	(0.050)	(0.047)	(0.284)	(0.095)
$\Phi^{\mathbb{Q}}$	-1.336	1.615	1.413	-0.130	0.548	0.251	-0.010	-0.029
	(0.235)	(0.329)	(0.085)	(0.087)	(0.011)	(0.015)	(0.010)	(0.010)
	-2.451	2.728	1.750	-0.059	-0.037	1.034	0.157	0.068
	(0.197)	(0.178)	(0.213)	(0.089)	(0.008)	(0.011)	(0.008)	(0.006)
	1.374	-0.876	0.550	0.046	0.043	0.050	1.396	0.035
	(0.217)	(0.157)	(0.195)	(0.068)	(0.044)	(0.066)	(0.045)	(0.037)
	4.635	-3.240	-2.961	1.613	0.014	0.022	0.058	1.217
	(0.297)	(0.458)	(0.251)	(0.169)	(0.018)	(0.026)	(0.016)	(0.015)

Table 5: Estimated conditional mean parameters under the $\mathbb{Q}$-measure. Standard errors in paranthesis.

(2014) emphasize that these term premia should be adjusted to correct for a small-sample bias; the resulting bias-corrected term premia are also shown in the figure denoted BRW.[9]

Considering the degree of mean-reversion in the factor dynamics implied the models, the GATSM and the Wright model stand out as two extremes: the former exhibit no mean-reversion and the latter is too mean reverting in the sense that it is affected by small-sample bias in the autoregressive parameter. In short, both estimates suffer from the persistence problem. The VTSM and BRW model represent in-between cases. These properties are clearly reflected in the implied term premia. In the GATSM, expectations of future yields are almost equal to current yields. In result, the expectations hypothesis is satisfied with highly stable and acyclical term premia. The Wright model, in contrast, attributes almost all variation in the forward rate to term premia. In fact, the correlation between the term premium and the forward rate is 0.96 as reported in Table 6. This implication of mean-reversion is also noted in Shiller (1979) and Kozicki & Tinsley (2001).

Both the VTSM and BRW term premia are stable and countercyclical. The cyclical patterns are most pronounced in the VTSM with very low premia in the years leading to

[9]The Wright and BRW term premia are both available at quarterly frequency from the website of Bauer et al. (2014). In both models, the factors are given by the first three principal components of the yield curve along with two unspanned macro risks constructed by smoothed inflation and GDP growth data.

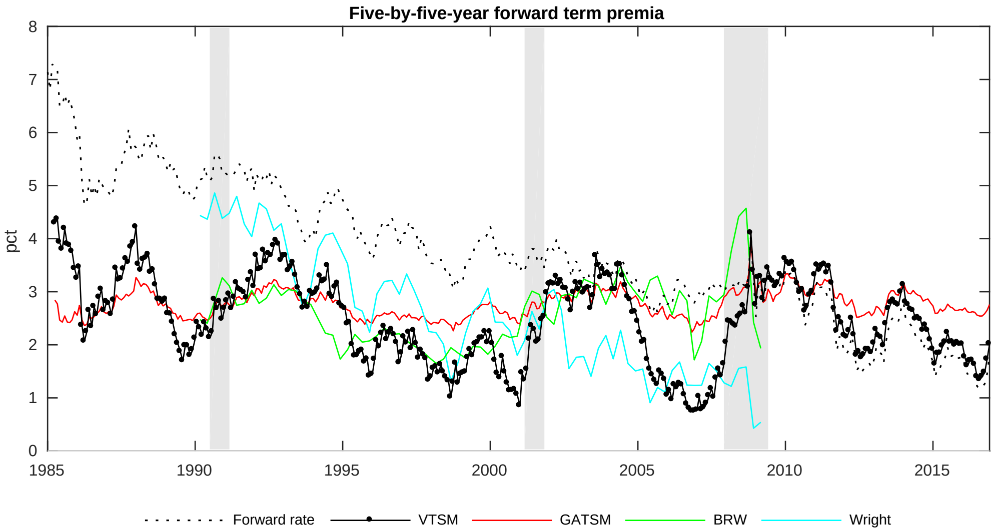

Figure 4: Five-by-five-year forward rate and corresponding term premia computed in the VTSM and GATSM. Forward term premia estimates from Bauer et al. (2014) and Wright (2011) are reported for reference (quarterly frequency). Shaded areas mark recessionary periods as defined by NBER.

the financial crisis of 2007/8. As reported in Table 6, the VTSM term premia are more volatile than the BRW, which shows that the volatility-induced stationarity introduces a higher degree of mean reversion than the affine model does after correcting for small-sample bias. The table illustrates further differences between the models: whereas the BRW premia are almost uncorrelated with the forward rate, the VTSM does imply some correlation. Thus, whereas the bias-corrected affine model predicts that the downward trend in long interest rates is caused by lower expected future short rates, the volatility-induced stationary model assigns this pattern partly to decreasing risk preferences and partly to the expectations component. As noted by Bernanke (2006), these differences carry different implications for the appropriate policy response. Thus, there are significant economic differences between the volatility-induced stationary model with reduced rank and the bias-corrected affine model.

Since term premia are unobserved, we can only argue that the VTSM appears more economically plausible compared to the GATSM. Therefore, I use survey data to support this conclusion with data and quantify the model differences. The data is from the Survey of Professional Forecasters (SPF) conducted by the Federal Reserve Bank of Philadelphia on a quarterly basis. I use median forecasts of the three-month Treasury bill rate as proxy for the short rate; the ten-year Treasury bond yield; and standardized CPI inflation following the construction of the inflation factor. I compare this data to

	VTSM	GATSM	BRW	Wright
Standard deviation	0.92	0.25	0.60	1.13
Correlation with forward rate	0.48	0.17	-0.09	0.96
Correlation matrix:				
VTSM	1			
GATSM	0.88	1		
BRW	0.44	0.63	1	
Wright	0.42	0.11	-0.10	1

Table 6: Standard deviations and correlations of term premia in the VTSM, the GATSM, Bauer et al. (2014) (BRW), and Wright (2011).

	VTSM			GATSM		
	Short rate	Long rate	Inflation	Short rate	Long rate	Inflation
3M	39.71	217.80	106.38	44.09	221.18	115.33
6M	49.14	200.23	98.49	58.74	205.46	113.78
1Y	74.56	167.48	92.66	83.90	174.33	106.66

Table 7: Root mean squared errors of model-implied expectations compared to survey data from Survey of Professional Forecasters. Reported in basis points.

expectations of the corresponding risk factors computed using respectively the VTSM and the GATSM. Table 7 compare root mean squared errors in basis points between model-implied and survey expectations for forecasting horizons of 3, 6, and 12 months ahead. The results unambiguously show that the VTSM matches market expectations better than the GATSM.

4.3 Out-of-Sample Performance

In this section, I show that volatility-induced stationarity improve the ability to forecast the yield curve. First, I estimate respectively factor dynamics (with one lag) and the parameters relating to the term structure cross-section in the VTSM and the GATSM for a sample from January 1985 to December 2005 (T=252). Then, using these estimated models I forecast the yield curve 3, 6, and 12 months ahead. I repeat this procedure by re-estimating the models based on a rolling-window sample from January 2006 to December

	VTSM			GATSM			Random Walk		
Horizon	3M	6M	12M	3M	6M	12M	3M	6M	12M
Average	**58.06**	**71.77**	**87.42**	69.07	88.21	102.74	63.37	79.48	125.61
1Y	65.90	**85.18**	**113.43**	99.57	131.34	168.83	**65.53**	102.71	175.08
2Y	**63.23**	**78.86**	**99.29**	86.35	111.69	136.38	64.60	91.40	156.03
3Y	**61.77**	**75.03**	**90.18**	77.96	99.04	115.84	64.20	83.29	140.76
4Y	**60.24**	**72.83**	**85.93**	72.16	90.75	103.14	63.82	78.08	129.29
5Y	**58.33**	**70.88**	**83.53**	67.52	84.74	94.88	63.45	75.12	120.86
6Y	**56.33**	**69.02**	**81.94**	63.50	79.94	89.03	63.12	73.64	114.60
7Y	**54.55**	**67.39**	**80.76**	59.96	75.87	84.56	62.80	73.00	109.85
8Y	**53.24**	**66.16**	**79.95**	56.90	72.36	80.96	62.47	72.73	106.11
9Y	**52.64**	**65.56**	**79.46**	54.35	69.37	78.05	62.07	72.54	103.05
10Y	54.36	**66.78**	79.72	**52.42**	66.95	**75.77**	61.60	72.28	100.44

Table 8: Root mean squared errors from forecasting the term structure using the VTSM and the GATSM estimated on a rolling window starting with the sample from January 1985 to December 2005. Forecasts by the random walk are reported for reference. The minimum value obtained for each forecast horizon and maturity is boldfaced. Reported in basis points.

2015. This period contains events that are difficult to forecast including the financial crisis of 2007/08 and the zero-lower bound regime.

Root mean squared errors from the exercise are presented in Table 8 along with random walk forecasts. The VTSM outperform both the GATSM and the random walk for all maturities but the 10-year yield. The differences between the models' forecasting performance increase inversely with maturity, which is consistent with the result that volatility-induced stationarity is generated by the short end of the yield curve.

4.4 In-Sample Performance

To compare the VTSM and the GATSM with respect to in-sample fit, Figure 5 shows the average observed and fitted yield curves for two years: 2006 and 2011. Year 2006 represents an expansionary period, where the conditional variance of the factors is estimated with a peak in the VTSM (see Figure 2). On average, the yield curve was nearly flat. In 2011, the conditional variances and yield levels were close to zero and the yield curve was on average upward-sloping. In both periods, the models match observed yields well

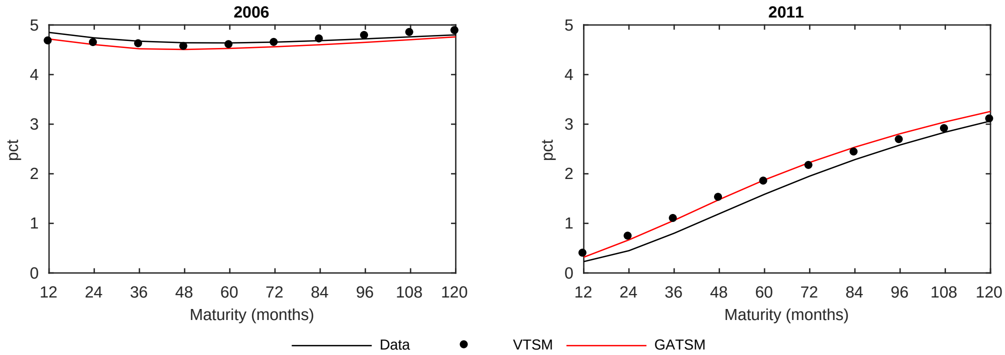

Figure 5: Observed and fitted yield curves averaged across 2006 (left) and 2011 (right).

and there are practically no difference between the models' capability of matching levels in sample.

I also consider the conditional second-order moments, which are time-varying in the VTSM and constant in the GATSM by construction of the factor dynamics. In the GATSM, $\text{var}_{t-1}(Y_{t,n}) = B_n' \Omega B_n$. For the VTSM, I apply a local linearization around X_{t-1} such that $\text{var}_{t-1}(Y_{t,n}) \approx [B_n' + C_n' g(X_{t-1})] \, \Omega_{t-1} \, [B_n' + C_n' g(X_{t-1})]'$, where the function $g(\cdot)$ is defined by $g(a) = \partial \text{vec}\,(X_t X_t') \,/ \partial X_t' |_{X_t = a}$. Since conditional variances are unobserved, I compare these measures to two proxies: realized variance computed from daily observations, also available from Gürkaynak et al. (2007), and rolling-sample variance with a 6-month look-back.

The results are shown for the three- and eight-year yields, which are representative for the entire term structure, in Figure 6. Considering the 3-year maturity, the VTSM captures the low conditional variance during the zero-lower bound regime with a peak centered at the outbreak of the financial crisis of 2007/08. Apart from these observations, the model-implied volatility appears to more correlated with levels of the risk factors than with the conditional volatility proxies. At the longer maturity, the level of the model-implied volatility is consistent with the proxies, but the fluctuations are not captured.

These results are supported by Filipović et al. (2017), who find evidence that yield volatilities are level-dependent near the zero-lower bound but unspanned at higher levels.[10] Also, Christensen et al. (2014) find that term structure models with spanned factors cannot capture realized yield volatility although they attribute this finding to that the realized volatility measure is misleading.

[10]Unspanned stochastic volatility is presented in Colin-Dufresne & Goldstein (2002).

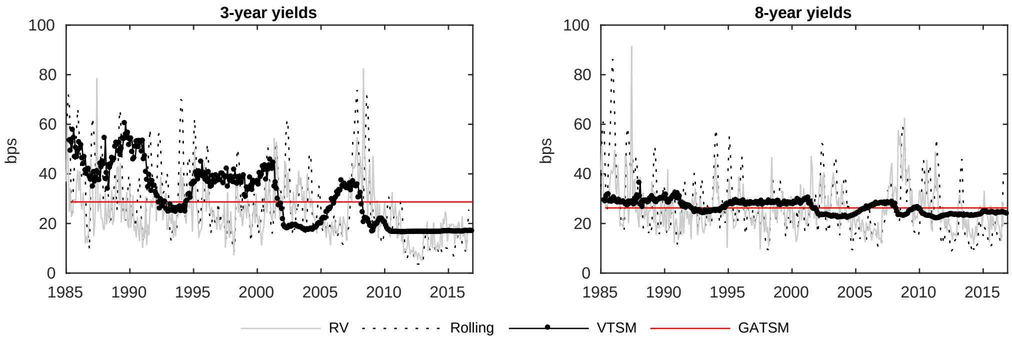

Figure 6: Conditional volatilities of 3- and 8-year yields implied by the VTSM and the GATSM. The former is computer by a local linearization. Proxies for conditional volatilities of the data given by respectively realized variances (RV) based on daily data and rolling-sample variance with a 6-month look-back are plotted for reference. Reported in basis points.

5 Concluding Remarks

I presented a volatility-induced stationary term structure model (VTSM), which allows interest rates to be stationary processes despite presence of unit roots and cointegration. The novelty of the model is that it exhibits volatility-induced stationarity, i.e., that the conditional variance depends on levels such that stochastic shocks affect levels more in periods where the process takes on large values in magnitude.

The combination of unit roots and mean reversion is important for estimating reliable term premia. In a nutshell, the proposed model provides a solution to the persistence problem of affine term structure models by estimating term premia that are time-varying but stable. In contrast, affine models imply term premia that are either almost constant (non-stationary case) or almost identical to interest rates (stationary case).

Bond yields in the VTSM can be approximated in closed form with sufficient accuracy. The empirical part of the paper presented a macro-finance model and identified volatility-induced stationarity in the short rate. Compared to the Gaussian affine term structure model with reduced rank, I showed that the VTSM obtains more realistic term premia and smaller out-of-sample forecasting errors.

Since the VTSM involves a time-varying conditional variance, one might expect the model to fit second-order moments of the yield curve. However, I find that further work is needed to fulfil this objective. In particular, examining the ability of the VTSM to match conditional yield variance may require unspanned factors of the conditional variance;

that second-order moments are targeted in estimation as in Cieslak & Povala (2016); or inclusion of derivatives in the information set as in Almeida et al. (2011), Bikbov & Chernov (2011) and Jagannathan et al. (2003). I leave these extensions for future research.

A Additional Tables

	$\mathcal{H}_0$	r_t	R_t	π_t	g_t
ADF test	unit root	2.16	3.19	-16.98	-14.83
		[0.44]	[0.17]	[0.00]	[0.01]
KPSS test	stationarity	1.49	1.89	0.39	0.43
		[0.00]	[0.00]	[0.08]	[0.06]

Table 9: Augmented Dickey-Fuller and KPSS tests for respectively unit root and stationarity. P-values in brackets.

	Bartlett	$r \leq 0$	$r \leq 1$	$r \leq 2$	$r \leq 3$
Trace test	No	90.89	32.62	15.99	5.06
		[0.00]	[0.02]	[0.04]	[0.02]
		{0.00}	{0.03}	{0.07}	{0.11}
Trace test	Yes	84.99	31.57	15.06	4.56
		[0.00]	[0.03]	[0.06]	[0.03]

Table 10: Johansen test of the null, $\mathcal{H}_0$: $r \leq r^\star < p$, against $r = p$ with and without Bartlett small-sample correction. P-values in brackets and bootstrapped p-values in curly brackets.

	VTSM				GATSM			
$\hat{\Sigma}$ ($\times 10^{-3}$)	0.045				0.369			
	(0.010)				(0.022)			
	0.049	0.217			0.029	0.228		
	(0.037)	(0.012)			(0.015)	(0.010)		
	-0.016	0.022	0.247		0.020	-0.007	0.275	
	(0.040)	(0.016)	(0.017)		(0.013)	(0.016)	(0.016)	
	0.012	0.014	0.002	0.113	0.013	0.017	-0.001	0.114
	(0.015)	(0.007)	(0.009)	(9,995)	(0.005)	(0.005)	(0.008)	(0.005)
$\hat{\Gamma}$	-0.096	-0.019	-0.015	-0.008				
	(0.013)	(0.006)	(0.006)	(0.005)				
	-0.015	0.007	0.012	-0.028				
	(0.014)	(0.010)	(0.011)	(0.012)				
	-0.006	-0.007	-0.012	0.076				
	(0.018)	(0.011)	(0.018)	(0.016)				
	-0.002	-0.000	-0.008	-0.0074				
	(0.006)	(0.004)	(0.007)	(0.007)				
$\hat{\mu}'$ ($\times 10^{-4}$)	-0.041	0.288	-0.273	-0.244	-0.637	-0.215	-0.122	-0.241
	(0.060)	(0.217)	(0.263)	(0.106)	(0.448)	(0.119)	(0.425)	(0.095)
$\hat{\Psi}_1$	-0.175	0.059	0.027	0.036	-0.427	0.183	0.014	-0.064
	(0.064)	(0.034)	(0.022)	(0.039)	(0.076)	(0.088)	(0.060)	(0.161)
	-0.026	0.027	0.150	0.140	-0.028	0.034	0.148	0.069
	(0.034)	(0.055)	(0.050)	(0.102)	(0.032)	(0.059)	(0.054)	(0.100)
	0.039	0.032	0.444	-0.087	0.020	0.024	0.449	0.029
	(0.032)	(0.056)	(0.059)	(0.113)	(0.030)	(0.056)	(0.077)	(0.140)
	0.027	0.010	0.068	0.244	0.020	0.021	0.065	0.236
	(0.016)	(0.027)	(0.022)	(0.048)	(0.015)	(0.027)	(0.023)	(0.049)
$\hat{\Psi}_2$	-0.028	-0.032	0.011	0.014	-0.125	0.164	0.035	0.527
	(0.072)	(0.023)	(0.019)	(0.045)	(0.082)	(0.080)	(0.065)	(0.165)
	-0.035	-0.119	0.064	0.048	-0.025	-0.117	0.057	0.005
	(0.040)	(0.055)	(0.044)	(0.106)	(0.039)	(0.053)	(0.048)	(0.114)
	0.036	-0.063	-0.162	0.292	0.015	-0.035	-0.124	0.480
	(0.032)	(0.049)	(0.066)	(0.118)	(0.033)	(0.052)	(0.079)	(0.137)
	0.008	-0.015	0.025	0.187	0.010	-0.009	0.017	0.193
	(0.017)	(0.025)	(0.029)	(0.053)	(0.016)	(0.025)	(0.033)	(0.049)
$\hat{\Psi}_3$	-0.061	-0.022	-0.0359	0.061	0.030	-0.064	-0.190	-0.070
	(0.056)	(0.022)	(0.0220)	(0.038)	(0.074)	(0.083)	(0.062)	(0.141)
	-0.045	0.065	-0.090	0.001	-0.026	0.056	-0.078	-0.063
	(0.034)	(0.058)	(0.043)	(0.096)	(0.034)	(0.061)	(0.044)	(0.099)
	0.007	0.064	0.122	-0.084	0.009	0.066	0.063	0.050
	(0.029)	(0.051)	(0.056)	(0.113)	(0.030)	(0.056)	(0.070)	(0.111)
	-0.039	-0.015	0.034	0.288	-0.035	-0.017	0.032	0.278
	(0.015)	(0.025)	(0.025)	(0.052)	(0.014)	(0.024)	(0.028)	(0.053)

Table 11: Short-run factor dynamics in the VTSM and GATSM. Standard errors in parentheses. $\hat{\Sigma}$ denotes the Cholesky decomposition of $\hat{\Omega}$.

	VTSM				GATSM			
	r_t	R_t	π_t	g_t	r_t	R_t	π_t	g_t
Ljung-Box test	7.71	1.67	0.61	4.96	3.72	1.48	1.05	5.75
	[0.10]	[0.80]	[0.96]	[0.29]	[0.44]	[0.83]	[0.90]	[0.22]
Engle's ARCH test	23.81	4.37	2.53	1.58	36.15	9.92	20.82	0.60
	[0.00]	[0.04]	[0.11]	[0.21]	[0.00]	[0.00]	[0.00]	[0.44]
Kolmogorov-Smirnov test	0.07	0.06	0.06	0.06	0.09	0.06	0.10	0.05
	[0.05]	[0.09]	[0.12]	[0.16]	[0.00]	[0.12]	[0.00]	[0.12]

Table 12: Residual specification tests: Ljung-Box test of no autocorrelation. Engle's test of no ARCH effects. Kolmogorov-Smirnov test of standard normal distribution. P-values in brackets.

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