Price discovery measures and High-frequency Data

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Abstract

For an asset traded in multiple venues, an outstanding problem is how those places individually contribute to the price discovery mechanism (the incorporation of information into prices). I show that existing measures of price discovery lead to misleading conclusions when using High-frequency data, due to uninformative microstructure noises. I then propose robust-to-noise measures, good at detecting "which market incorporates quickly new information". Using the Dow Jones stocks traded on NYSE and NASDAQ on the period March 1st to May 30th 2011, I show that the data are in line with my theoretical conclusions. In addition, when the Information Share measure gives wide bounds making it unusable, my proposed robust IS has very close bounds. I later obtain that price discovery mostly happens on NYSE and is positively correlated with its liquidity and its market share in small and big size transactions. For NASDAQ-listed stocks, large quantities trades do not convey information and NASDAQ contribution to price discovery increases slightly the days with macroeconomic announcements.

Keywords: *Price discovery, Information Share, Permanent-Transitory component, Microstructure noise, Realized Variance*

JEL: *C32, C58, G14*

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1 Introduction

The institutional evolutions of financial markets and the development of High-frequency Trading generated a growing literature on the resulting consequences on market's outcomes. The multiplication of trading platforms coupled with the internationalization of financial markets resulted in some assets being listed simultaneously in many town or even many countries. Similarly traders can send orders in remotely located market places. The trading prices for a given security on those interrelated markets are strongly linked by arbitrage activities. A similar situation arises for one security and its derivatives: The spot prices are related to futures prices, the CDS prices are related to the credit spread.

The price discovery mechanism is generally understood as the process by which information is computed into prices, it is interesting in the multiple markets setup to understand how each market does so. An international investor for instance, choosing how to split the orders in different markets, might find it worthy to know where the price is close to the fundamental. The regulator also, in its quest to the best market organization, is interested in which market contributes to the price movement of an asset and for which reasons[1]. This quest of the market with the "best" information processing mechanism goes back to Garbade and Silber (1983) problem: which market is dominant and which market is satellite?

To determine in which market price discovery happens some tools (known as price discovery measures) are developed in the literature. Hasbrouck (1995) pioneer paper, using the Beveridge-Nelson permanent component, presented a measure called the Information Share (IS) and provided comparison of NYSE and regional exchanges in the quotes formation of Dow stocks. The main competing measure to Hasbrouck (1995) is the PT measure inHarris et al. (2002b), consisting of the common factor weight in the permanent-transitory (PT) decomposition of Gonzalo and Granger (1995). Those measures are intensively debated by De Jong (2002), Lehmann (2002), Hasbrouck (2002),Baillie et al. (2002), and Yan and Zivot (2010). One conclusion of the debate is that the IS accounts more for the variability in the price discovery process and the permanent (efficient) price identified by Hasbrouck (1995) has an economic relevance[2].

The other part of the debate lies in their view of price discovery. Hasbrouck (1995) sees it as *"who moves first"* in the process of price adjustment and Harris et al. (2002b) as the process by which security markets attempt to identify permanent changes in equilibrium transaction prices. Meanwhile, what their proposed measures actually capture is unclear. And as stated by Lehmann (2002), a market should dominate the price discovery if it is the best in incorporating information

[1] Eun and Sabherwal (2003) report that the Canadian authority was really worried about US-markets becoming the place where the Canadian's stock prices were computed

[2] The PT relies on a permanent price that is not a random walk

in a "timely and efficient" manner. This widely accepted characterization of market dominance presents two dimensions. The first dimension is the timing: a market reflecting quickly new information is close to efficiency. The second dimension is the avoidance of noises. A market with less noises is also efficiently incorporating information. The noises can come from uninformative sources as bid-ask bounce, price discreteness, and measurement errors. It is then not very clear which dimension is actually captured by the existing measures. For example, using Monte Carlo exercises, Putniņš (2013) obtains that IS and PT are actually assessing how markets avoid noises. Whereas Yan and Zivot (2010) obtains in a specific structural VECM that the PT assess how markets avoid noises while the IS captures both dimensions.

This study innovates in exposing new facts on price discovery measures, particularly linked to the utilization of High-frequency data. Using those data bring issues that are studied in the literature for volatility estimation in the presence of microstructure noises (see Andersen et al., 2000; Zhang, 2010; Jacod et al., 2009). I show that IS and PT are not related to the fundamental value but rather to information-uncorrelated noises. This could lead to misleading interpretations in applications. I also contribute to the literature by proposing new measures that are robust-to-noise and restore a clear interpretation of what is being measured: My robust IS (ISR) and robust PT (PTR) measures are good at detecting which market incorporates *quickly* new information. My framework incidentally provides values to compare the pure noise in the markets.

If both "speed" and "noise-avoidance" dimensions of price discovery are relevant and meaningful, confusions might come in utilization of price discovery measures as their nature can change given the frequency of data at hand. The analysis of price discovery should disentangle the previous two dimensions for the following reasons:

First, the way most papers consider a market to be informationally dominant is that, once new information is available, the price of the asset on this market is the first to reflect it. But this market might be more affected by information uncorrelated-noise, if it has a different tick size for example. It is thus unclear which effect will dominate in the measure or which market reveals more about the fundamental value. Let's take the extreme case where one market's price equals the efficient price plus a noise with infinite variance, and another market's price is the one-period lagged efficient price. The latter market is clearly more informative about the efficient price even if the first market is the fastest. It thus appears that another source of confusion about what the measures will do is the size of the noise in the data. On this matter, I provide analytical insights on how price discovery measures are related to microstructure noises and the sampling frequency.

Secondly, Hasbrouck (1995) defines its price discovery measure as the contribution of a market's innovation to the variance of the innovation in the efficient price. He then suggests that his Information Share is good at detecting which market moves first. This statement is somewhat giving more importance to the fact that a market is the first to incorporate information. In addition, the IS has an identification problem and is only able to produce bounds[3]. Sometimes, bounds can be wide making the IS useless. Hasbrouck (1995) recommendeds to sample at High-frequency to reduce the correlation and tighten the IS bounds, but this practice ignores that at High-frequency non informative part of the noise dominates the variances estimation[4]. Meanwhile in application, Chakravarty et al. (2004) use IS and are interested in the timing sequence when they justify their contribution to the literature by stating: "there is surprisingly little evidence that new information is reflected in option prices *before* stock prices". My paper emphasizes that at high frequency the IS is not related to the efficient price and rather measures which market avoids noise.

Lastly, an endogeneity problem could arise in a number of applications. The values provided by price discovery are used as dependent variables in regression to investigate the determinants of a market's dominance. Chakravarty et al. (2004) use IS to show that 17% of informed trading happens in the options markets, and that price discovery across strike price is determined by relative spread, leverage, and volume. Huang (2002) uses the IS to compare who has the most timely and informative quote, between Electronics communication Networks (ECNs) and Nasdaq; they find that measures of market liquidity do not necessarily explain the market maker's contribution to price innovation. Barclay et al. (2003) study the impact of trading costs variables on the Information Share of ECNs. Eun and Sabherwal (2003) regress the PT coefficients on the relative spread, volume, listing age, and market Cap, to explain the contribution of Toronto Stock Exchange (TSE) and U.S. exchanges to price discovery of cross-listed Canadian stocks. As an example, in Chakravarty et al. (2004), the price discovery of the option market, measured by the IS, tends to be greater when the effective bid-ask spread is narrow relative to the stock market. If by definition the IS were to fully capture the bid-ask spread noise, then there is full endogeneity in their regression of the IS on the bid-ask spread. By disentangling the two aspects of price discovery, my proposed robust measures can be used to avoid the endogeneity issue.

In the application, using data of NYSE TAQ database, I examine if my conclusions are in line with the data. I observe that indeed the data seem to present the patterns I highlighted, but the frequency of the transactions might not be high enough to show certain features. As quotes data are more frequent, I do the same analysis with mid-quotes of some assets and it confirms my theoretical conclusions. I then investigate the relative contribution of NYSE and NASDAQ to the price formation of Dow Jones assets. The robust IS measure performs well as it has very close bounds, when the standard IS bounds are wide and thus unusable. Descriptively, NYSE captures the big part of volume traded but NASDAQ is the most liquid with a high level of activity. This implies

[3] it is based on the Cholesky decomposition of variance matrix and is thus dependent of variables ordering.

[4] This is related to the signature plot of Andersen et al. (2000)

that NASDAQ mostly runs the orders of small quantities while NYSE runs big quantities orders. In terms of contribution to price discovery for the assets under investigation, NYSE is generally dominant. The contribution of a market appears to be positively correlated with its liquidity. I also analyze the correlation between market's contribution and markets share in each category of trade size. It reveals that the contribution of a market is correlated with its share in small size transactions. For NASDAQ listed stocks, there is no correlation with market's share in big size transactions, so large quantities trades do not convey information.

The Macroeconomic indicators[5] announcements are some of the times where fundamental information arrive in the markets, I compute the measures for the News-days and for the non-News days. I obtain that the contribution of NASDAQ to price discovery slightly increases the days with news. Traders wanting to exploit quickly those public pre-scheduled news, could prefer to do so on the most liquid market.

The remainder of the paper is organized as follow. The second section reviews the main existing measures of price discovery. The third section presents some structural microstructure models and the price discovery measures are analytically computed at High-frequency. In the fourth section I propose the robust-to-noise measures and present their performances in some simulation exercises. In the fifth section, an application is done on assets of the Dow Jones that are listed and traded on NYSE and NASDAQ on the period March 1st to May 30th 2011.

2 Measuring price discovery

Constructing a price discovery measure would normally require that the object of interest be clearly identified. There is a current and permanent discussion in this respect with existing measures. This originates from the fact that they are defined on a reduced form model and not in a structural model. The approach to build prices discovery measures is to extract a common unobserved permanent price from the observed prices, and to attribute its characteristics to each market.

Let's consider an asset traded on markets 1 and 2 at the respective prices p_{1t} and p_{2t}[6]. This is done via the VECM representation of the cointegrated price vector $p_t = (p_{1t}, p_{2t})'$. The gap between the two prices $(p_{1t} - p_{2t})$ is stationary such that there exists only one common trend for the prices. In fact, because the prices in the two markets are from the same asset, a gap between them can not remain infinitely as there will be room for profits by arbitrage (for example buying continuously in the first market and selling in the second). Under the previous notations and restrictions implied by arbitrage, Johansen (1991) results imply that the price vector admits the following

[5] E.g: Non farm payroll, Fed Fund rates, CPI, PPI...

[6] The results are easily obtained for more than 2 markets

Vector Error Correction Model (VECM):

$$\Delta p_t = -\alpha\beta' p_{t-1} + \Gamma_1 \Delta p_{t-1} + \ldots + \Gamma_K \Delta p_{t-K} + e_t, \tag{1}$$

where the cointegrating matrix is $\beta' = \begin{pmatrix} 1 & -1 \end{pmatrix}$ as $\beta' p_t = p_{1t} - p_{2t}$ is stationary. e_t is an independent white noise with $var(e_t) = \Omega$.

The Granger representation theorem gives the following transformation of 1 where $\Psi(L)$ is a lag polynomial:

$$p_t = p_0 + \Psi(1)\sum_{s=1}^{t} e_s + \Psi^*(L)e_t, \tag{2}$$

where the matrix $\Psi(1)$ is given by

$$\Psi(1) = \beta_\perp \left(\alpha_\perp^{'} \left(I - \sum_{i=1}^{p} \Gamma_i \right) \beta_\perp \right)^{-1} \alpha_\perp^{'}. \tag{3}$$

The representation 2 entails a decomposition of the prices in a stationary component $p_0 + \Psi^*(L)e_t$ and a permanent component $\Psi(1)\sum_{s=1}^{t} e_s$. The matrix $\Psi(1)$ summarizes the long run impact of the innovation e_t on prices p_t .

2.1 The Information Share measure

Hasbrouck (1995) looks for a measure that will determine on which market the price discovery does happen. He proposes to use the contribution of each market to the variance of the innovation of the "efficient price" price.

As $\beta' = \begin{pmatrix} 1 & -1 \end{pmatrix}$, its orthogonal $\beta_\perp^{'} = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and the formula 3can be written as

$$\Psi(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \psi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} \psi_{11} & \psi_{12} \end{pmatrix}.$$

The 2×1 row ψ replaced in equation 2 yields

$$p_t = p_0 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \psi \sum_{s=1}^{t} e_s + \Psi^*(L)e_t. \tag{4}$$

This representation displays a scalar random walk component of the prices $\psi\sum_{s=1}^{t} e_s$, and a stationary part $\Psi^*(L)e_t$ that might be attributed to transitory effects. The common permanent component is identified as the implicit fundamental price of the asset. Something to notice here is that e_t drives both the permanent and the transitory component. So, the construction does not distinguish the non-informative noise (due for example to tick size or measurement errors) from the information-correlated frictions that would be due to information asymmetry, market under/over reaction (Menkveld et al., 2007).

The new information entering the fundamental price is the innovation ψe_t, and its variance $(\psi\Omega\psi')$ is the total Information Share. Hasbrouck (1995) defines the market contribution to price discovery in the following way.

If Ω is diagonal, then the total Information Share is

$$\psi\Omega\psi' = \psi_{11}^2\Omega_{11} + \psi_{22}^2\Omega_{22}$$

and the Information Share (IS) for the market j, defined as the relative contribution of this market in the variance of the new information, is obtained as:

$$IS_1 = \frac{\psi_1^2\Omega_{11}}{\psi_{11}^2\Omega_{11} + \psi_{22}^2\Omega_{22}} \quad \text{and} \quad IS_2 = \frac{\psi_2^2\Omega_{22}}{\psi_{11}^2\Omega_{11} + \psi_{22}^2\Omega_{22}}. \tag{5}$$

As Ω is not diagonal in general, Hasbrouck (1995) suggests using its Cholesky root to obtain a lower triangular matrix F, such that $\Omega = FF'$. An identification problem arises as the ranking of the variables matters for the Cholesky decomposition. That is the matrix F changes with the ordering of the variables in the prices vector. Thus, the Information Share measure can only provides an upper and a lower bounds.

When the market 1 is placed in the first position in p_t, then $\Omega = FF'$, and I have the bounds

$$IS_{u,1} = \frac{([\psi F]_1)^2}{\psi\Omega\psi'} \quad \text{and} \quad IS_{l,2} = \frac{([\psi F]_2)^2}{\psi\Omega\psi'}, \tag{6}$$

where $[\psi F]_j$ represents the jth element of the vector ψF.

Now if the market 1 is switched to the 2nd position in p_t, the new Cholesky root $\tilde{F}$ is obtained such that $\Omega = \tilde{F}\tilde{F}'$. The others bounds are

$$IS_{u,2} = \frac{([\psi\tilde{F}]_1)^2}{\psi\Omega\psi'} \quad \text{and} \quad IS_{l,1} = \frac{([\psi\tilde{F}]_2)^2}{\psi\Omega\psi'} \tag{7}$$

The non-uniqueness of the Information share is a problem for applications as the measure are used as dependent variable in regression. Many studies thus, simply consider the lower bound or take the mid-bounds (see Chakravarty et al., 2004; Putniņš, 2013).

The IS identification issue is related to the Macroeconomics VAR literature problem of identifying the structural shocks from the reduced form model. Relying on Hasbrouck (1995)'s efficient price, some authors tried to solve this by doing some transformations of the innovation variance matrix. The limit of those techniques is that they completely lose an economic meaning behind the mathematical operations. For example, Lien and Shrestha (2014) use an orthogonalization of the correlation matrix to propose a measure that is independent of the variables ordering. Meanwhile there is no economic intuition behind the orthogonalization of the correlation matrix. Grammig and Peter (2013) exploit "tail dependence" for identification which is done through heteroskedasticity on two regimes as in Rigobon (2003), Lanne and Lütkepohl (2010). The drawback is that identification relies on the data and it is not always the case that they provide enough tail dependence to identify unique information share. Another limit of all the existing method based on Hasbrouck (1995) efficient price is that they lack a testing theory. This is not the case of the PT measure, which in turn, has the severe drawback that its efficient price is not a random walk.

2.2 The Permanent-Transitory measure

The main competitor to IS is the Gonzalo and Granger (1995) common factor weight in the Permanent-Transitory (PT) decomposition. This consists of decomposing a difference stationary time series as the sum of a permanent component Q_t and a transitory stationary component T_t. The identification of the two components of $p_t = Q_t + T_t$ relies on two assumptions:

- T_t does not Granger-cause Q_t in the long run,
- T_t is a linear combination of the observed variables.

In the context of one asset and many markets, the permanent component is driven by a difference stationary[7] factor (f_t) that is common to both markets, such that the observed prices vector can be written as

$$p_t = \begin{bmatrix} 1 \\ 1 \end{bmatrix} f_t + T_t.$$

The common factor is a linear combination of current prices $f_t = \gamma_1 p_{1t} + \gamma_2 p_{2t}$. It is easily shown that given the ECM equation 1, the weight $(\gamma_1 \gamma_2)$ are proportional to $\alpha_\perp$ such that:

$$f_t = c\alpha_{1\perp} \times p_{1t} + c\alpha_{2\perp} \times p_{2t}$$

with c constant.

Harris et al. (2002a) evaluate the relative contribution to price discovery of market 1 and market 2 by taking the weight of each market in the permanent component as

[7] Or integrated of order 1 denoted $I(1)$

$$PT_1 = \frac{\alpha_{1\perp}}{\alpha_{1\perp}+\alpha_{2\perp}}\ , PT_2 = \frac{\alpha_{2\perp}}{\alpha_{1\perp}+\alpha_{2\perp}}.$$

The link between the permanent price extracted by Hasbrouck (1995) and the permanent price of Harris et al. (2002b) is studied by De Jong (2002). A difference between the PT measure with the IS measure is that f_t is a linear combination of only the current prices. Thus the permanent component of the Gonzalo and Granger (1995) decomposition is generally not a random walk. This is a serious limitation as this permanent component could not represent an efficient price and only gets an economic meaning in a structural model (see Lehmann, 2002). Baillie et al. (2002) show that IS and PT can be computed easily after the estimation of the VECM and they present the relationship linking PT to IS. In the case of a diagonal Ω, the PT squared coefficients are weighted by the innovations variances to obtain IS. This is seen by deducing $\begin{pmatrix} \psi_{11} & \psi_{12} \end{pmatrix} = c \begin{pmatrix} \alpha_{1\perp} & \alpha_{2\perp} \end{pmatrix}$ from formula 3 and replacing for example in formulas 5 to obtain $IS_1 = \left(\alpha_{1\perp}^2 \Omega_{11}\right) / \left(\alpha_{1\perp}^2 \Omega_{11} + \alpha_{2\perp}^2 \Omega_{22}\right)$.

Instead of focusing on the innovation variance, the permanent component share relies on the error correction weighting matrix $\alpha_{\perp}$. In this respect Eun and Sabherwal (2003) also think of price discovery as the adjustment to the equilibrium and access it by the coefficient α summarizing how a market corrects a departure from the other market price. Building the measures with only a coefficient of the VECM allows those methods to have testable implications and thus test of statistical significance can be performed.

3 Microstructure models and sampling frequency

The price discovery measures presented in Section 2 are used in the literature to detect which model is likely to have generated the observed log prices $p_t \equiv \left(p_{1t}, p_{2t}\right)'$. Are the two markets structurally identical? Is one market leading the information while the other is lagged? To compare the performances of the measures in answering those questions, literature relies on some structural microstructure models (see Hasbrouck, 2002; Harris et al., 2002a) representing the different situations that might arise on market . I rewrite versions of those models to make them dependent of the sampling interval h and a delay parameters δ. For those models, Δp_t generally admits a Vector Moving-Average of order 1 ($VMA(1)$) representation, allowing to compute analytically the values of the prices discovery metrics. The VMA(1) equation is

$$\Delta p_{th} = e_{th} + \Theta e_{th-h} \text{ with } \Theta = \begin{pmatrix} -1+c & 1+d \\ c & d \end{pmatrix}, \tag{8}$$

where e_{th} is the white noise innovation with variance $\Omega = var\left(e_{th}\right) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$.

The long run impact matrix is thus

$$\Psi(1) = I + \Theta = \begin{pmatrix} c & 1+d \\ c & 1+d \end{pmatrix} = \psi \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ with } \psi = \begin{pmatrix} c & 1+d \end{pmatrix} \tag{9}$$

To compute the measures, one needs the values of the parameters Ω, c, d in terms of the structural parameters in p_t. For this, the values of the structural variance and autocovariance are matched with the ones of the VMA(1) equation 8. That is

$$\begin{aligned} C_0 = var(\Delta p_{th}) &= \Omega + \Theta\Omega\Theta' \\ C_1 = cov(\Delta p_{th}, \Delta p_{th-h}) &= \Theta\Omega \end{aligned} \tag{10}$$

Computing ΘC_0 and replacing $\Theta\Omega$ by C_1 gives the equation 11

$$C_1 - \Theta C_0 + \Theta C_1 \Theta' = 0 \tag{11}$$

For each of the model I will present, I computed Θ by solving this matrix equation via long and tedious calculations given in appendix, and then Ω is obtained as $\Omega = \Theta^{-1} C_1$. Next, I present the structural models of interest and study the behavior of the price discovery measures.

3.1 Model I: A two-market "Roll" model.

In model I, both markets incorporate the efficient price m_t. This situation could arise from markets with no private information, and an efficient price driven by public non-traded information. At the sampling interval h, the latent fundamental log price of the asset is

$$m_{th} = m_{th-h} + \eta_{th}$$

The innovation is $\eta_{th} = \sigma_h \mathcal{N}(0,1)$ and its variance $\sigma^2(h)$ converges to zero when h goes to zero. This is not a limitation as empirically the returns and their variance become very small at high frequency. It can also be viewed in the discretization of the often-used continuous time model $dm_t = \sigma dB_t$, implying $\sigma_h = \sigma\sqrt{h}$. The observed prices are contaminated by i.i.d non correlated microstructure noises

$$\begin{aligned} p_{1th} &= m_{th} + c_1\varepsilon_{1th} \\ p_{2th} &= m_{th} + c_2\varepsilon_{2th} \end{aligned} \tag{12}$$

$\varepsilon_{1t}, \varepsilon_{2t}, \sim \mathcal{N}(0,1)$ with $E(\varepsilon_{1t}\varepsilon_{2t}) = E(\eta_{th}\varepsilon_{1t}) = E(\eta_{th}\varepsilon_{2t}) = 0$. The constants c_1, c_2 represent the variances of the noise components. They could be made dependent of h and going to zero but less faster than σ_h. This will not change the main conclusions as all the facts I describe remain qualitatively the same.

In this setup, there is no market dominating the price discovery process considered as the predominance in incorporating the new information η_{th}.

I compute the variance and covariance 10 and obtain

$$C_0 = \begin{pmatrix} \sigma_h^2 + 2c_1^2 & \sigma_h^2 \\ \sigma_h^2 & \sigma_h^2 + 2c_2^2 \end{pmatrix} \text{ and } C_1 = \begin{pmatrix} -c_1^2 & 0 \\ 0 & -c_2^2 \end{pmatrix}.$$

Using C_0 and C_1 to solve equation 11, the values of ψ and Ω that are necessary to compute the different measures, are obtained in terms of the structural parameters σ_h^2, c_1^2, c_2^2.

Lemma 1. *In model I, solving equations 44 gives*

$$\psi = \kappa \begin{pmatrix} c_1^{-2} & c_2^{-2} \end{pmatrix} \quad \textit{and} \quad \Omega = K \begin{pmatrix} c_1^2\left(1 - c_2^{-2}\kappa\right) & \kappa \\ \kappa & c_2^2\left(1 - c_1^{-2}\kappa\right) \end{pmatrix} \tag{13}$$

$$\text{With} \quad \kappa = -\frac{1}{2}h\sigma^2 + \sqrt{h}\sigma\frac{1}{2}\sqrt{h\sigma^2 + \frac{4c_1^2c_2^2}{\left(c_1^2 + c_2^2\right)}}, \tag{14}$$

$$K = \left[1 - \kappa\left(c_1^{-2} + c_2^{-2}\right)\right]^{-1}. \tag{15}$$

Proposition 1. *The PT measure*

Using the results in Lemma 1, the PT gives

$$PT_1 = \frac{c_1^{-2}}{c_1^{-2} + c_2^{-2}} \textit{ and } PT_2 = \frac{c_2^{-2}}{c_1^{-2} + c_2^{-2}}. \tag{16}$$

Proof. See Appendix □

The PT does not assess the priority to incorporate m_t but is completely dependent of noises. The contribution of a market is inversely proportional to its own noise. That is, the market with the lowest noise has the biggest contribution, and the PT is measuring the avoidance of noises at any frequency. It is only when the level of noise is the same in the two markets, that the measure can be coherently interpreted in term of fundamental information with an equal value for each market.

Proposition 2. *The IS measure,*

Using the results in Lemma 1, The IS bounds for market 1 and for market 2 are computed as

$$IS_{u,1} = \frac{c_1^{-2}}{\left(c_1^{-2}+c_2^{-2}\right)\left(1-c_2^{-2}\kappa\right)} \text{ and } IS_{l,1} = \frac{c_1^{-2}K^{-1}}{\left(c_1^{-2}+c_2^{-2}\right)\left(1-c_1^{-2}\kappa\right)}, \tag{17}$$

$$IS_{u,2} = \frac{c_2^{-2}}{\left(c_1^{-2}+c_2^{-2}\right)\left(1-c_1^{-2}\kappa\right)} \text{ and } IS_{l,2} = \frac{c_2^{-2}K^{-1}}{\left(c_1^{-2}+c_2^{-2}\right)\left(1-c_2^{-2}\kappa\right)}. \tag{18}$$

At High-frequency (when $h \simeq 0$), the parameter $\kappa \simeq 0$ and

$$IS_{u,1} \simeq IS_{l,1} \to \frac{c_1^{-2}}{c_1^{-2}+c_2^{-2}} = PT_1, \tag{19}$$

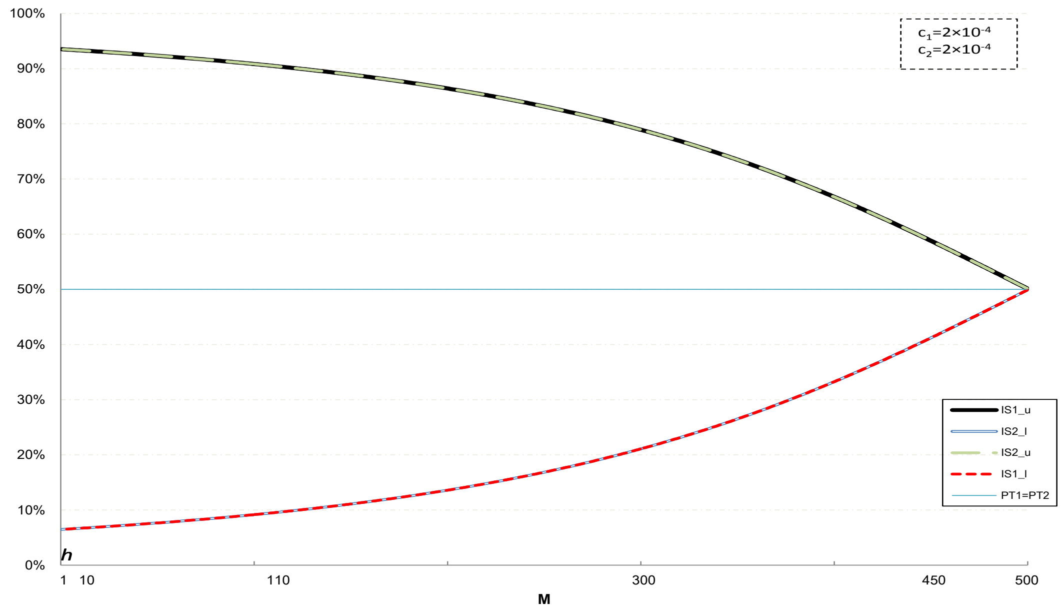
$$IS_{u,2} \simeq IS_{l,2} \to \frac{c_2^{-2}}{c_1^{-2}+c_2^{-2}} = PT_2. \tag{20}$$

Proof. *See Appendix* □

In this model at high frequency, when h is small, the bounds on the Information Share becomes tighter and close to the value of the PT measure. But the limiting values are dominated by information-uncorrelated microstructure noises and are not related to the fundamental value. This result challenges the interpretation of price discovery measure in term of the fundamental price. At high frequency, the parameter $\sigma^2$of the fundamental price disappears from the formulas and we are let with a comparison of the level of noises. So if c_1^2 is smaller than c_2^2, then $IS_1 = PT_1 > PT_2$, and ones might conclude that the Market 1 is fast to compound new information, while the market are actually equally fast. The formulas 19 and 20 could meanwhile be taken as positive result, in the sense that they provide items to compare the costs of trading in different markets for a cross listed asset.

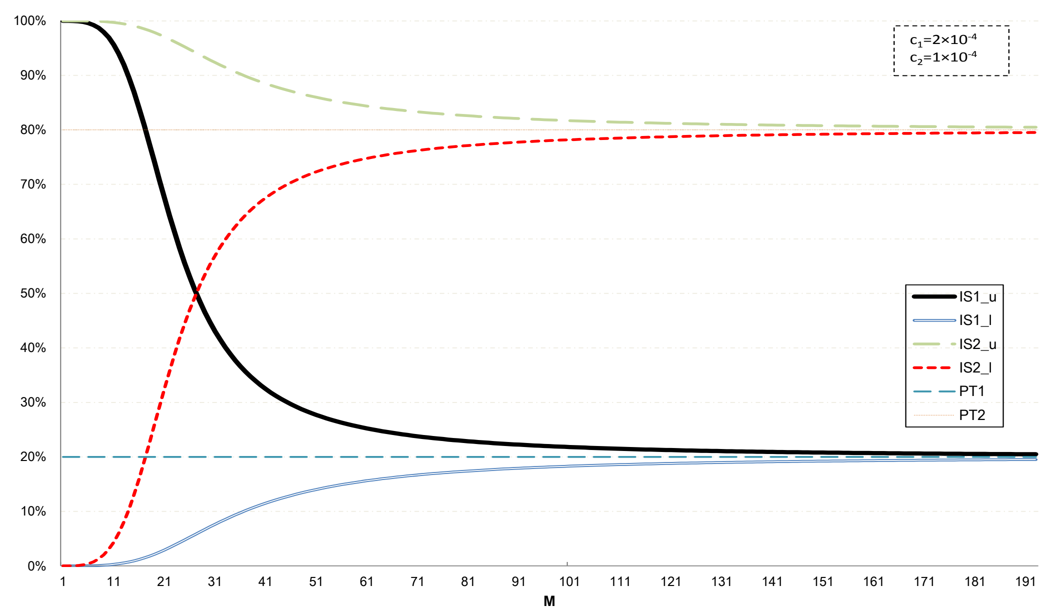
To explore how the measures depend on the noise and the frequency we plot the IS and PT as a function of $M = 1/h$. In Figure 1 with equal level of noise the bounds are wide at lower frequency but go to 50% when the frequency ($M = 1/h$) increases. When the level of noise is different (Figure 2) the bounds are reduced but then the market with the smallest noise becomes dominant.

Figure 1: Equal noises $c_1^2 = c_2^2$



Note: The figures plot the IS and the PT measures computed analytical on model I. The horizontal axis represents the sampling frequency $M = 1/h$.

Figure 2: Different noises $c_1^2 = 2c_2^2$



Note: The figures plot the IS and the PT measures computed analytical on model I. The horizontal axis represents the sampling frequency $M = 1/h$.

3.1.1 Time Varying noises

The previous results are derived under constant noises variances. The following theorem show that all the conclusions remain when the noises variance vary with the sampling frequency as long as the fundamental return decreases less faster than h.

Proposition 3. *Time Varying noises: Let* $c_1^2 \equiv c_1'^2 h^{\alpha_1}$, $c_2^2 \equiv c_2'^2 h^{\alpha_2}$ *with* $\alpha_1, \alpha_2 > 0$

If $Max(\alpha_1, \alpha_2) < 1$ *, then*

$$\begin{cases} \kappa = & -\frac{1}{2} h\sigma^2 + \sqrt{h}\frac{\sigma}{2}\sqrt{\sigma^2 h + \frac{4c_1'^2 c_2'^2 h^{\alpha_1} h^{\alpha_2}}{\left(c_1'^2 h^{\alpha_1} + c_2'^2 h^{\alpha_2}\right)}} \longrightarrow 0 \\ K = & \left[1 - \kappa\left(c_1'^{-2} h^{-\alpha_1} + c_2'^{-2} h^{-\alpha_2}\right)\right]^{-1} \longrightarrow 1 \end{cases}$$

And

$$IS_{u,i} \simeq IS_{l,i} \overset{h\to 0}{\longrightarrow} PT_i = \frac{c_i'^{-2}}{c_1'^{-2} + c_2'^{-2}}, \, i = 1,2$$

Proof. See appendix □

3.2 Model II: The Roll model with a delayed market

The fundamental log price of the asset is still driven by the innovation $\eta_{th} = \sigma_h \mathscr{N}(0,1)$ with $\sigma(h) = \sigma\sqrt{h}$, and

$$m_{th} = m_{th-h} + \eta_{th}.$$

The first market incorporates m_t, but the second market is delayed of δ. The observe prices are

$$\begin{aligned} p_{1th} =& m_{th} + c_1\varepsilon_{1th} \\ p_{2th} =& m_{th-\delta} + c_2\varepsilon_{2th}. \end{aligned} \tag{21}$$

I compute the variance and covariance 10 as

$$C_0 = \begin{pmatrix} h\sigma^2 + 2c_1^2 & (h-\delta)\sigma^2 \\ (h-\delta)\sigma^2 & h\sigma^2 + 2c_2^2 \end{pmatrix} \text{ and } C_1 = \begin{pmatrix} -c_1^2 & 0 \\ \delta\sigma^2 & -c_2^2 \end{pmatrix}.$$

When $h > \delta$ the price admits a VMA(1) representation and I calculate the analytical solutions by solving the matrix equation 11

Proposition 4. *The PT measure*

In Model II, solving equations 44 gives

$$
\begin{aligned}
PT_1 &= \frac{\left(-\frac{1}{2}\frac{\sigma^2\left[\left(\delta\sigma^2+c_2^2\right)\left(\sigma^2\delta(h-\delta)+h\left(c_1^2+c_2^2\right)\right)+2c_1^2c_2^2\delta\right]}{c_1^2\left[\left(\delta\sigma^2\right)^2+\delta\sigma^2c_2^2+c_1^2c_2^2+c_2^4\right]}\pm\frac{1}{2}\sqrt{\Delta}\right)}{\left(-\frac{1}{2}\frac{\sigma^2\left[\left(\delta\sigma^2+c_2^2\right)\left(\sigma^2\delta(h-\delta)+h\left(c_1^2+c_2^2\right)\right)+2c_1^2c_2^2\delta\right]}{c_1^2\left[\left(\delta\sigma^2\right)^2+\delta\sigma^2c_2^2+c_1^2c_2^2+c_2^4\right]}\pm\frac{1}{2}\sqrt{\Delta}\right)\left(1+\frac{c_1^2}{\delta\sigma^2+c_2^2}\right)-\frac{\delta\sigma^2}{\delta\sigma^2+c_2^2}} \\
PT_2 &= 1-PT_1
\end{aligned}
$$

where

$$
\Delta = \left[\frac{\sigma^2\left[\left(\delta\sigma^2+c_2^2\right)\left(\sigma^2\delta\left(h-\delta\right)+h\left(c_1^2+c_2^2\right)\right)+2c_1^2c_2^2\delta\right]}{c_1^2\left[\left(\delta\sigma^2\right)^2+\delta\sigma^2c_2^2+c_1^2c_2^2+c_2^4\right]}\right]^2+4\frac{c_2^2\sigma^2\left[\sigma^2\delta\left(h-\delta\right)+hc_2^2\right]}{c_1^2\left[\left(\delta\sigma^2\right)^2+\delta\sigma^2c_2^2+c_1^2c_2^2+c_2^4\right]}
$$

Proof. See appendix □

The formula for the IS, which is also very cumbersome, is computed after obtaining Ω by the equation 36. The formulas are not really intuitive, but it displays the fact that PT depend on the information parameter σ, on the frequency parameter h, and on the delay δ. Here, the limit when h is small can not be easily obtained analytically. In fact, by computing the autocavariance for the process for $h < \delta$, the order of the VMA becomes bigger than 1 and increases when h decreases. We will rely on graphical analysis for more insights.

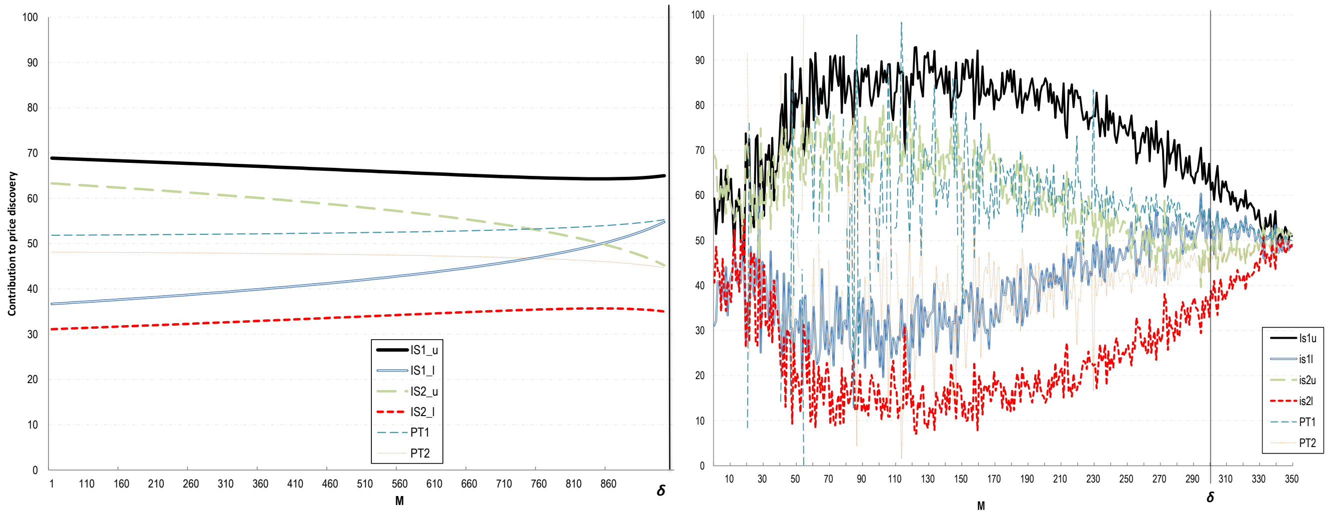
The behavior of the measures in Model II is summarized in Figures 3a and 3b. The Panel A of the graph corresponds to $h > \delta$ is plotted using the analytical formulas. The Panel B is plotted for all h using simulations. In this setup by construction, the first market dominates structurally the price discovery mechanism as it is the first to compute new information. When the level of noise are equals (Figure 3a), the measures succeed in designing market 1 as dominant when $h \geq \delta$. But at high frequency with $h < \delta$, the measures converges to 0.5, stating that the two markets are equally contributing to the price discovery mechanism. When the market 1 is noisier than market 2 (Figure 3b), the measures in both panels seem to converge to values such that market 2 is dominant. Theses results simply reflect the relative size of noise in market 1, compared to noise in market 2.

3.3 Model III: A Two-market model with public and private information

In this model presented by Hasbrouck (2002), the efficient price is driven by informative trading on the market 1 (η_{1th}) and a non-traded public information $\eta_{th} = \sigma_h \mathcal{N}(0,1)$. The dynamic of price is

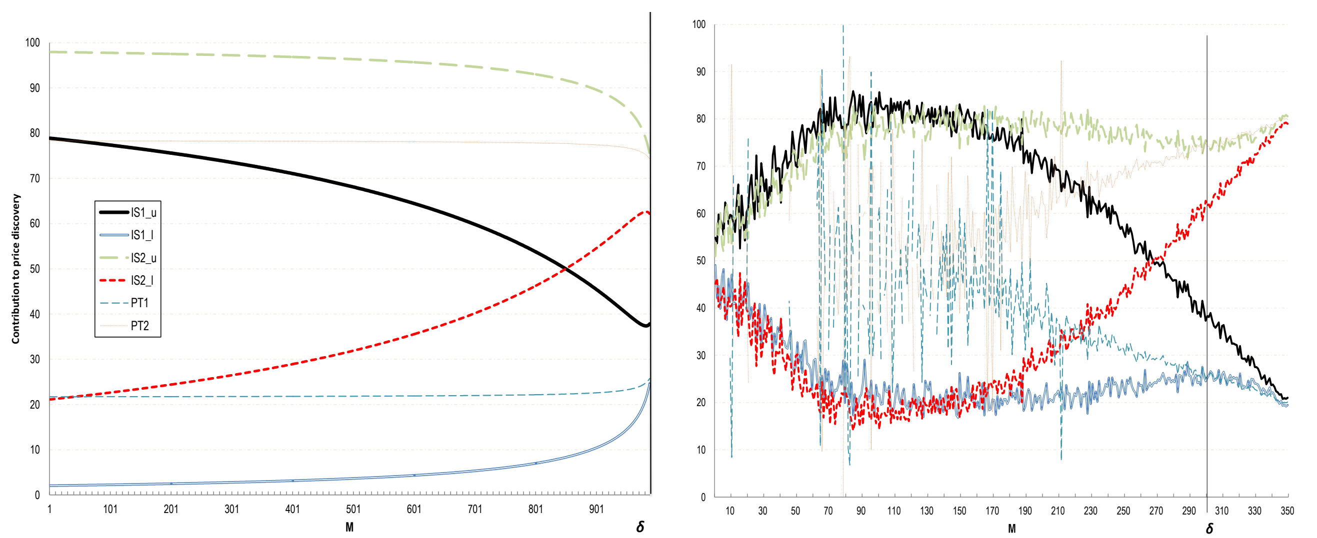
Figure 3: Model II: performances of the measures

(a) Equal noises



Panel A Panel B

(b) : Different noises



Panel A Panel B

Note: The figure plots the IS and PT model II. The horizontal axis represents the sampling frequency $M = 1/h$. *Panel A and Panel B are separated at the point where* $h < \delta$. $c_2^2 = 0.002/2$.

described by the following system

$$\begin{aligned} m_t =& m_{t-h} + \lambda_h \eta_{1th} + \eta_{th} \\ p_{1th} =& m_{th} + \eta_{1th} + c_1 \varepsilon_{1th} \\ p_{2th} =& m_{th-h} + c_2 \varepsilon_{2th} \end{aligned} \tag{22}$$

where λ_h, the liquidity parameter, goes to zero with h for the same reasons as σ_h in the previous sections. The Market 2 relies on a delayed value (with lag h) of m_t. The parts of microstructure noises that are information-uncorrelated are ε_{1th} and ε_{2th}.

As before, Market 1 is dominant from the structural point of view.

We solve for the equation 11 and the results at the order of $\sqrt{\lambda_h}$ are :

Proposition 5. *The PT Share*

In Model III, the solutions of equations 44 at the order of $\sqrt{\lambda_h}$ gives

$$PT_1 = \frac{1}{1 + c_2^{-2}\left(1 + c_1^2\right)} + o\left(\sqrt{\lambda_h}\right) \quad \textit{and} \quad PT_2 = \frac{c_2^{-2}\left(1 + c_1^2\right)}{1 + c_2^{-2}\left(1 + c_1^2\right)} + o\left(\sqrt{\lambda_h}\right)$$

Proof. See appendix □

Proposition 6. *The IS bounds*

In Model III, the solutions of equations 44 at the order of $\sqrt{\lambda_h}$ gives

$$\begin{array}{rcll} IS_{u,1} & = & -\frac{1}{c_2^{-2}\left(1+c_1^2\right)D-1} \times \frac{1}{1+c_2^{-2}\left(1+c_1^2\right)} & +o\left(\sqrt{\lambda_h}\right) \\ IS_{l,1} & = & \frac{D\left(1+c_2^{-2}\left(1+c_1^2\right)\right)-1}{D-1} \times \frac{1}{1+c_2^{-2}\left(1+c_1^2\right)} & +o\left(\sqrt{\lambda_h}\right) \\ IS_{u,2} & = & \frac{-\left(1+c_1^2\right)c_2^{-2}}{(-1+D)} \times \frac{1}{1+c_2^{-2}\left(1+c_1^2\right)} & +o\left(\sqrt{\lambda_h}\right) \\ IS_{l,2} & = & \frac{D\left(1+c_2^{-2}\left(1+c_1^2\right)\right)-1}{Dc_2^{-2}\left(1+c_1^2\right)-1} \times \frac{c_2^{-2}\left(1+c_1^2\right)}{1+c_2^{-2}\left(1+c_1^2\right)} & +o\left(\sqrt{\lambda_h}\right) \end{array}$$

with $D = \sqrt{\lambda_h}\left(\sqrt{\left(1+c_1^2\right)\left(1+c_2^{-2}\left(1+c_1^2\right)\right)}\right)^{-1} \xrightarrow{h\to 0}$.

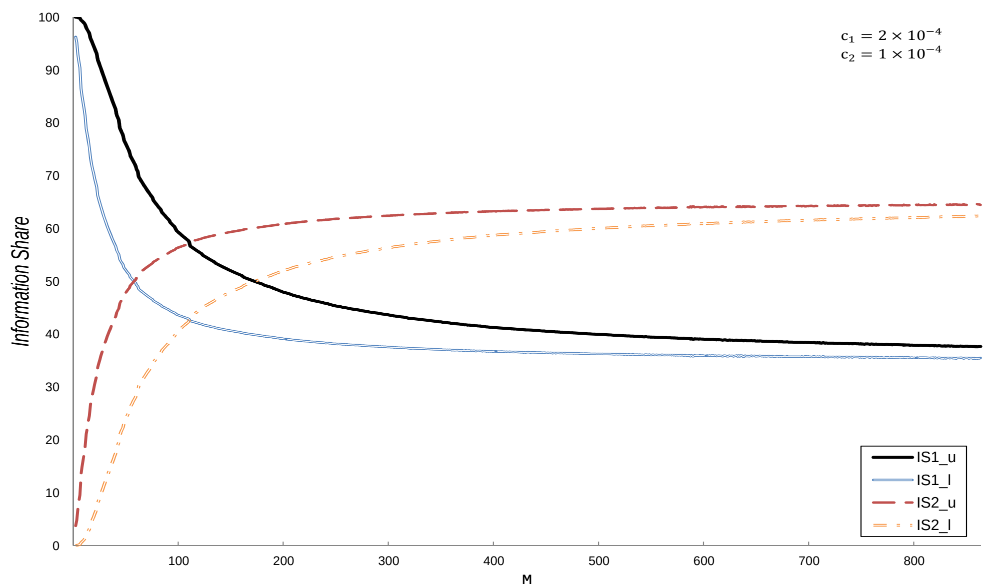
When h

$$\begin{array}{rcccl} IS_{u,1} \simeq IS_{l,1} \simeq PT_1 & \longrightarrow & \frac{1}{1+c_2^{-2}\left(1+c_1^2\right)} & = & \frac{c_1^{-2}}{c_1^2c_2^{-2}+c_1^{-2}+c_2^{-2}} \\ IS_{u,2} \simeq IS_{l,2} \simeq PT_2 & \longrightarrow & \frac{\left(1+c_1^2\right)c_2^{-2}}{1+c_2^{-2}\left(1+c_1^2\right)} & = & \frac{c_2^{-2}\left(c_1^2+1\right)}{c_1^2c_2^{-2}+c_1^{-2}+c_2^{-2}} \end{array}$$

Proof. See appendix □

When h is small in this setup, IS and PT give the same value. The contribution of market 2 decreases with the noise variance in market 2, and increases with the noise variance in market 1. When the level of noise is equal in the two markets, $PT_2 > PT_1$ and market 2 is chosen as the dominant market, which is in opposition with the structural model. In Figure 4, I compare the measures for the model III computed numerically for h decreasing. Even if the model is changing by reducing the delay parameter $\delta = h$, market 1 remains dominant as it drives the efficient price. At lower frequency, the IS of market 1 is almost 100% and the IS of market 2 is close to 0, even if market 1 is the noisiest. When the values of h is small, the contribution of market 2 is bigger than that of market 1, suggesting falsely that market 2 is dominant. The issues highlighted here are less important with small noises variances or with small noises difference between the two markets. The frequency at which the dominance commutes increases (see Figures A.2a and A.2b in appendix).

Figure 4: Model III: IS with sampling frequency and delay



Note: The figure plots the IS model III. The horizontal axis represents $M = 1/h$. The PT (not plotted here) has the same pattern

Remark: By fixing $\delta = 0$ and $\eta_{th} = 0.$, we obtain $m_t = m_{t-h} + \lambda_h \eta_{1th}$ and

$$\begin{aligned} p_{1th} &= m_{th} + \eta_{1th} + c_1 \varepsilon_{1th} \\ p_{2th} &= m_{th} + c_2 \varepsilon_{2th} \end{aligned} \tag{23}$$

corresponding to a Two-market model with overreaction. It is not very clear which market dominates the price discovery in this setup. The market 1 incorporates m_t but there is an overreaction to information in the observed prices. The market 2 also incorporates timely m_t. The computation using the formula 32 in appendix gives directly PT as

$$PT_1 = \frac{c_1^{-2}}{c_1^2 c_2^{-2}\left(\lambda_h \sigma^2 + 1\right) + c_1^{-2} + c_2^{-2}} \quad \text{and} \quad PT_2 = \frac{c_2^{-2}\left(c_1^2\left(\lambda_h \sigma^2 + 1\right) + 1\right)}{c_1^2 c_2^{-2}\left(\lambda_h \sigma^2 + 1\right) + c_1^{-2} + c_2^{-2}}$$

We still see that the measures vary inversely proportional to noises. But the contribution of market 2 increases with the variance of the efficient price. For equal level of information uncorrelated-noises, market 2 has a greater contribution than market 1. So the PT captures the "efficient" facet of prices.

All the new facts just explained here are warnings about the interpretations when using existing price discovery measures on High-Frequency data. It is thus of interest to develop a price discovery measure that is adapted in high frequency data and clarify what aspect of the market is actually captured.

4 Robust-to-Noise price discovery measures

The different analytical computations showed that at High-frequency the price discovery measures are dominated by noises. In this sense, the measures seem to be better interpreted in terms of noises avoidance. This point is also made by Yan and Zivot (2010) who suggest combining IS and PT in one measure to reduce the noises effects. The issue here is related to the debate in the literature about the property that those price discovery measures are actually capturing. My results suggest that at lower frequency they might be capturing the speed at which markets incorporate information while at a high frequency they are capturing which market is less noisy. This creates a misleading interpretation caused only by the frequency of observations. To restore a consistency in the definition of the measures at all frequency, I propose a correction of the measures to reduce the effect of noises. For this, a look at the different formulas suggests that the measures are dominated by a factor equal to the inverse variance of the market microstructure noises. I thus propose to robustify the IS and the PT by multiplying them by the noise variance.

So the bounds on IS and the PT for the market 1 are multiplied by c_1^2, and the bounds on IS and PT for the market 2 are multiplied by c_2^2 . I re-normalize the robust to-noise versions of the measures to keep the sum to one:

$$ISR_{u,1} = \frac{c_1^2 IS_{u,1}}{c_1^2 IS_{u,1} + c_2^2 IS_{l,2}} \quad \text{and} \quad ISR_{l,1} = \frac{c_1^2 IS_{l,1}}{c_1^2 IS_{l,1} + c_2^2 IS_{u,2}} \tag{24}$$

$$ISR_{u,2} = \frac{c_2^2 IS_{u,2}}{c_1^2 IS_{l,1} + c_2^2 IS_{u,2}} \quad \text{and} \quad ISR_{l,2} = \frac{c_2^2 IS_{l,2}}{c_1^2 IS_{u,1} + c_2^2 IS_{l,2}} \tag{25}$$

$$PTR_1 = \frac{c_1^2 PT_1}{c_1^2 PT_1 + c_2^2 PT_2} \quad \text{and} \quad PTR_2 = \frac{c_2^2 PT_2}{c_1^2 PT_1 + c_2^2 PT_2} \tag{26}$$

Obviously, if Ω is diagonal, here too for each market, I have equality of its lower and upper bounds ($ISR_{u,1} = ISR_{l,1}$ and $ISR_{u,2} = ISR_{l,2}$).

To compute the previous quantities, estimations of the microstructure noise variances c_1^2 and c_2^2 are required. Fortunately, the literature on integrated volatility estimation in the presence of microstructure noises provides good ones. At High-frequency, the realized volatility (the sum of squared log return) divided by $(2n)$ is a good approximation of the noise variance (see Andersen et al., 2000; Zhang, 2010; Jacod et al., 2009). I thus consider the estimators

$$\widehat{c_1^2} = (2n)^{-1} \sum_{t=1}^{n} \Delta p_{1t}^2 \text{ and } \widehat{c_2^2} = (2n)^{-1} \sum_{t=1}^{n} \Delta p_{2t}^2$$

The properties of this estimator of the noise variance are proven in Zhang (2010). The intuition behind the results is the following. Let's Consider an observed price written as $p_{th} = m_{th} + c_0 \varepsilon_{th}$ with $\varepsilon_{th} \sim i.i.d\, \mathcal{N}(0,1)$ and $\Delta m_{th} = \eta_{th} = \sigma_h \mathcal{N}(0,1)$, $E(\eta_{th}\varepsilon_{th}) = 0$, the variance of the intraday return $\left(\sigma_h = O(\sqrt{h})\right)$ decreases with the sampling interval $h = 1/n$. The expectation of the realized volatility is

$$\begin{aligned} E\left(\sum_{t=1}^{n} \Delta p_t^2\right) &= \sum_{t=1}^{n} E\left(\eta_{th}^2 + c_0^2 \Delta \varepsilon_{th}^2 + 2c_0 \eta_{th} \Delta \varepsilon_{th}\right) \\ &= n\sigma_h^2 + 2n \times c_0^2 \\ &= O(nh) + 2n \times c_0^2 \\ &\simeq 2n \times c_0^2 \end{aligned}$$

This development incidentally provides a way to evaluate the noise in the data. In fact, if one is to consider only how markets avoid noises, the values of c_1^2 and c_2^2 estimated previously could measure price discovery in the sense of "which market is not noisy".

4.1 Simulation

We analyze through Monte Carlo simulations the performances of the robust measures (ISR and PTR) relatively to IS and PT. For this, I simulate the structural models I, II and III (12,21, 22). For each model, I simulate a sample of 23 400 observations (to imitate a trading day in second), then the data are sampled at a given frequency (1, 2, 5, 10, 60) and the measures are computed in a VECM. The order of the VECM is chosen using the Akaike Information Criterion (AIC) which is typically what people do in practice. Each design is replicated 1000 times and the numbers in Tables 1-3 are the averages results and standard deviations (in parenthesis) over the 1000 replications. The gray-shaded columns of the Tables are the robust-to-noise estimates. The results are presented only for market 1.

In model I, both markets incorporate m_t, so a good estimate of price discovery in term of "where information enter the price first" should be 0.5. When the two markets have the same level of information-uncorrelated noises (Table 1, Panel A), all the measures perform well with values close to 1/2. When the market 1 is noisier than market 2 (see Table 1, Panel B), the estimated mid-bounds on IS and the PT (0.35 and 0.33) are far below 0.5 when data are sampled at High-frequency ($h = 1$). Meanwhile, my proposed robust mid-bounds ISR is 0.48 and PTR is 0.47, suggesting rightly that both markets are similar in incorporating m_t. All the metrics perform quite well at lower frequency.

In model II, the market 2 is slow and incorporates new information with a lag $\delta = 3s$, and market 1 noise's variance is set to $c_1^2 \equiv 0.0002$, bigger than $c_2^2 \equiv 0.0001$ of market 2. Price discovery happens in market 1, but the small values in Table 2, obtained for IS and PT falsely suggests that it happens in market 2. By using the robust measures the good interpretation is restored with values for ISR and PTR close to 0.89 and 0.76. The effect is more pronounced in Table 3 where the first market drives the fundamental price. While the other measures suggest an equal role for both markets in the price discovery process, the robust-to-noise measures are almost 0.99. The estimated values presented here depend on the size of the noise and on the sampling frequency. The performances of IS and PT are improved when the noise is reduced or when the difference in noises between the two markets diminishes. But the qualitative result remains unchanged: the robust measures are better than IS and PT to detect which market incorporates timely new information.

5 Empirical application

I study the daily relative part in the price discovery of assets of the Dow Jones Industrial Index that are listed and traded on NYSE and NASDAQ. I focus on the trade prices coming from the TAQ Database and covering the period from the 01 March to the 30 May 2011. Before using the

Table 1: Simulation Results: Model I

Panel A: $c_1^2 = c_2^2$								
h	$IS_{u,1}$	$IS_{l,1}$	$ISR_{u,1}$	$ISR_{l,1}$	IS_1	ISR_1	PT_1	PTR_1
1s	0.65	0.35	0.50	0.50	0.50	0.50	0.50	0.50
	(0.06)	(0.06)	(0.04)	(0.08)	(0.06)	(0.06)	(0.04)	(0.04)
5s	0.77	0.22	0.50	0.49	0.50	0.50	0.49	0.49
	(0.12)	(0.11)	(0.07)	(0.22)	(0.11)	(0.15)	(0.14)	(0.14)
10s	0.83	0.17	0.50	0.50	0.50	0.50	0.50	0.50
	(0.14)	(0.14)	(0.08)	(0.31)	(0.14)	(0.20)	(0.28)	(0.28)
Panel B: $c_1^2 = 2c_2^2$								
h	$IS_{u,1}$	$IS_{l,1}$	$ISR_{u,1}$	$ISR_{l,1}$	IS_1	ISR_1	PT_1	PTR_1
1s	0.51	0.19	0.54	0.42	0.35	0.48	0.33	0.47
	(0.06)	(0.05)	(0.05)	(0.09)	(0.06)	(0.06)	(0.04)	(0.05)
5s	0.7	0.11	0.54	0.36	0.41	0.45	0.34	0.42
	(0.13)	(0.08)	(0.07)	(0.23)	(0.1)	(0.15)	(0.15)	(0.17)
10s	0.78	0.09	0.53	0.36	0.44	0.44	0.34	0.37
	(0.15)	(0.11)	(0.08)	(0.32)	(0.13)	(0.2)	(0.3)	(0.34)
30s	0.82	0.14	0.51	0.45	0.48	0.48	0.33	0.26
	(0.2)	(0.17)	(0.1)	(0.38)	(0.18)	(0.19)	(2.76)	(2.02)

Estimates for market 1 of the Information Share (IS) bounds, the PT share, the robust ISR and PTR. It is computed on simulated prices of Model I:

$$\begin{array}{lll} m_{th} = m_{th-h} + \eta_{th}, & \eta_{th} \backsim \sigma_h \mathcal{N}(0,1), \sigma_h = T^{-0.5}, & panel\, A:\ c_1^2 = c_2^2 = 2.10^{-4} \\ p_{ith} = m_{th} + c_i \varepsilon_{ith}, & \varepsilon_{ith} \sim \mathcal{N}(0,1), i = 1,2, & panel\, B:\ c_1^2 = 2c_2^2 = 2.10^{-4} \end{array}$$

A path of T=23400 observations is generated, prices are sampled at each interval h and a VECM is estimated with lag chosen by AIC. The values presented are the averages and the standard deviation (in parenthesis) over 1000 simulated paths.The gray shaded columns are robust measures. The reference value is 0.50

Table 2: Simulation Results: Model II

h	$IS_{u,1}$	$IS_{l,1}$	$ISR_{u,1}$	$ISR_{l,1}$	IS_1	ISR_1	PT_1	PTR_1
1s	0.39	0.29	0.90	0.88	0.34	0.89	0.16	0.76
	(0.06)	(0.06)	(0.02)	(0.03)	(0.06)	(0.03)	(0.02)	(0.03)
2s	0.33	0.19	0.84	0.77	0.26	0.81	0.14	0.69
	(0.09)	(0.07)	(0.05)	(0.10)	(0.08)	(0.08)	(0.03)	(0.06)
3s	0.30	0.14	0.80	0.66	0.22	0.73	0.13	0.63
	(0.10)	(0.08)	(0.08)	(0.17)	(0.09)	(0.13)	(0.04)	(0.13)
5s	0.30	(0.10	0.74	0.51	0.20	0.63	0.12	0.53
	(0.12	(0.08)	(0.11)	(0.23)	(0.10)	(0.17)	(0.06)	(0.24)
10s	0.34	0.08	0.67	0.36	0.21	0.52	0.11	0.40
	(0.18)	(0.10)	(0.16)	(0.28)	(0.14)	(0.22)	(0.11)	(1.77)

The Table reports estimates for market 1 of the Information Share (IS) bounds, the PT share, the robust ISR and PTR). It is computed on simulated prices of Model II: $m_{th} = m_{th-h} + \eta_{th}$, $p_{1th} = m_{th} + c_1\varepsilon_{1th}, p_{2th} = m_{th-\delta} + c_2\varepsilon_{2th}$, $\varepsilon_{ith}, (\eta_{th}/\sigma_h) \sim \mathcal{N}(0,1), i = 1,2$, $\sigma_h = T^{-0.5}, c_1^2 = 0.002$, $c_2^2 = 0.0001$, $\delta = 3$. *A path of T=23400 observations is generated, prices are sampled at each interval h and a VECM is estimated with lag chosen by AIC. The values presented averages and standard deviations (in parenthesis) over 1000 simulated paths.The gray shaded columns are robust measures. The reference value is 1.*

Table 3: Simulation Results: Model III

h	$IS_{u,1}$	$IS_{l,1}$	$ISR_{u,1}$	$ISR_{l,1}$	IS_1	ISR_1	PT_1	PTR_1
1s	0.47	0.49	1.00	1.00	0.48	1	0.01	1
	(0.19)	(0.19)	(0.00)	(0.00)	(0.19)	(0.00)	(0.01)	(0.00)
2s	0.25	0.26	1.00	1.00	0.26	1	0.01	0.99
	(0.20)	(0.20)	(0.03)	(0.04)	(0.20)	(0.03)	(0.06)	(0.02)
3s	0.22	0.22	0.99	0.99	0.22	0.99	0.01	1.00
	(0.21)	(0.21)	(0.09)	(0.09)	(0.21)	(0.09)	(0.04)	(0.51)
5s	0.18	0.18	0.98	0.98	0.18	0.98	0.01	0.61
	(0.22)	(0.22)	(0.10)	(0.10)	(0.22)	(0.10)	(0.03)	(11.57)
10s	0.24	0.23	0.98	0.97	0.24	0.98	0.00	0.99
	(0.27)	(0.27)	(0.13)	(0.09)	(0.27)	(0.11)	(0.26)	(0.21)

The Table reports estimates for market 1 of the Information Share bounds (ISu,ISl), the robust IS bounds (ISlr,ISur), the PT and the robust PT (PTr). It is computed on simulated prices of Model III: $m_{th} = m_{th-h} + \lambda_h \eta_{1th}$, $p_{1th} = m_{th} + \eta_{1th} + c_1 \varepsilon_{1th}$, $p_{2th} = m_{th-\delta} + c_2 \varepsilon_{2th}$, $\varepsilon_{ith}, (\eta_{th}/\sigma_h) \sim \mathcal{N}(0,1), i = 1,2$, $\sigma_h = \lambda_h = T^{-0.5}, c_1^2 = 0.002$, $c_2^2 = 0.0001$, $\delta = 3$. *A path of T=23400 observations is generated, prices are sampled at each interval h and a VECM is estimated with lag chosen by AIC. The values presented averages and standard deviations (in parenthesis) over 1000 simulated paths.The gray shaded columns are robust measures. The reference value is 1.*

data a cleaning job is done on the raw data: First, I suppress the data stamped before the opening (9h30) and after the closing (16h00) of the market. I also remove the data between 9h35 because the activity at the opening session creates a lot large values with respect to the daily continuous activity I aim to study. Second, to handle the synchronicity problem, I fill the data with the last trade price.

5.1 Descriptive analysis

The Dow Jones stocks data, on NYSE and NASDAQ, amount to 30 assets on a 3 month period for a total of 22,444,752 observations. NYSE and NASDAQ are the two biggest exchanges in the world by capitalization and trade value. NYSE remains by far the first with a capitalization of around 14 USD trillion in 2011 (around 16 USD trillion in 2014). During this year, the trade value was about 20 USD trillions, which represents an average daily amount of 55 USD billions. NASDAQ has a market capitalization of 4.6 USD trillions, and a trade value of 13.5 USD trillions, corresponding to an average daily amount of 37 USD billions[8].

Concerning where the assets are traded, the domination is not that pronounced as shown by the average daily statistics in Table B.1 in appendix. For JP Morgan (JMP) for example, around 5 millions of share are traded each day on NYSE, while 4.8 millions are traded on NASDAQ. This pattern is the same for most of the stocks, that is to say that NYSE concentrates the biggest part of share exchanged in a day. For few assets like PFE and GE, NASDAQ dominates the exchanges in term of volume. If we look at the liquidity (we think of liquidity as the frequency of transactions), NASDAQ dominates for almost all assets. For PFE we have around 16,153 trading times in one day on NASDAQ, while we have only 7 080 trading times on NYSE. This is not in contradiction with the analysis of volumes, it simply states that most of the trades of bigger size happens on NYSE, while NASDAQ is characterized by a lot of trades of small quantities (details in table B.2 in appendix). For example, NASDAQ cumulates 43.3% of small size trades for American express (AXP) and only 23.4% of big size trades, while NYSE cumulates 57.3% of big size trades.

Those descriptive statistics also show that, if prior-belief is that price discovery is completely driven by the liquidity or by the volume of share traded, the answer is not straightforward as we have for each market depending on the asset: high-volume and high-liquidity, high-volume and low-liquidity, low-volume and high-liquidity.

[8] Source: (http://www.i3investor.com/jsp/hti/usmarket.jsp).

5.2 Results on markets contribution

Before looking at market dominance, I compute the IS and the PT measures for the assets at different sampling frequency with the VECM-lag chosen by AIC. I obtain the same type of patterns described in Section 3 with the structural models. Figures A.2 and A.3 plot the results for American Express (AXP) and Exxon Mobil Corporation (XOM). It shows that the evolution of the measures with sampling frequency looks like the theoretical path up to a given frequency. It doesn't show the crossing of the lines, but this might be just that raw data are not frequent high enough to display all the interesting features. We can only have convincing guess by looking at the limit of the lines. In fact, for most of the stocks, the number of transactions per day is such that the interval h is between $4s$ and $7s$.

Now, let's consider the mid-quotes data at the microsecond frequency for Microsoft and Pfizer on the December 12th 2013. This choices are imposed only by data accessibility, and we consider NYSE ARca (market 1) and NASDAQ (market 2). This trading day corresponds to an amount of 424,876 observations for Microsoft, and 149090 for Pfizer. Figure A.4 shows the results of the IS and the PT measures with respect to the sampling frequency. It confirms that the interpretation of the results can change with the sampling frequency, and that the IS bounds tighten to the same values at high frequency.

The results on markets' contributions in Table 4 show that, for most of the stocks, NYSE appears to be the dominant market. The dominance of NYSE on NASDAQ is strong for MMM, NKE and TRV. NASDAQ dominates the price discovery mechanism for BA, CAT, GS, IBM. The table 4 also reports the lower and the upper bounds on the IS. It shows that bounds are quite wide for all assets (for example NYSE has a 28% to 85% contribution for American Express-AXP) which clearly complicates the interpretation. Meanwhile the robust IS indicates that the contribution is between 55% and 56% coherent with the numbers for PT and PTR. These results also indicates that the markets' structure have really changed during the recent years. For comparison, in Hasbrouck (1995), NYSE concentrated most of the trades resulting in more than 90 % of the contribution to price discovery.

We now compute the correlation of each market's contribution to price discovery with its share in different categories of transaction size. We see that (table 5) for all the exchanges the correlation between their contribution and their market share in small-size transactions is 1/3. The correlation with big-size trades is 0.27 for NYSE-listed share, while it is only -0.04 for the set of NASDAQ-listed shares. The correlation of the ISR with the liquidity does not show a specific pattern. In summary, price discovery happens generally on NYSE for the stocks under investigation, the contribution of a market is correlated with its market share for small and medium size transactions. In the results, The robust IS measures present another advantage over the IS. When the bounds on IS

Table 4: Contribution to price discovery of the NYSE

	$IS_{u,1}$	$IS_{l,1}$	$ISR_{u,1}$	$ISR_{u,1}$	IS_1	ISR_1	PT_1	PTR_1
NYSE-listed stocks								
AXP	0.85	0.28	0.55	0.56	0.59	0.67	0.60	0.63
BA	0.64	0.14	0.36	0.39	0.47	0.33	0.39	0.40
CAT	0.79	0.21	0.48	0.50	0.54	0.52	0.51	0.53
CVX	0.89	0.31	0.63	0.60	0.57	0.72	0.63	0.65
DD	0.82	0.28	0.54	0.55	0.57	0.63	0.58	0.60
DIS	0.89	0.36	0.63	0.63	0.63	0.79	0.67	0.71
GE	0.79	0.36	0.53	0.58	0.64	0.71	0.62	0.68
GS	0.73	0.23	0.47	0.48	0.50	0.47	0.49	0.49
HD	0.88	0.33	0.59	0.60	0.62	0.75	0.64	0.69
IBM	0.74	0.25	0.48	0.49	0.53	0.51	0.51	0.52
JNJ	0.91	0.41	0.66	0.66	0.67	0.84	0.71	0.75
JPM	0.89	0.29	0.59	0.59	0.61	0.75	0.64	0.68
KO	0.87	0.29	0.58	0.58	0.60	0.72	0.63	0.66
MCD	0.87	0.40	0.63	0.63	0.62	0.75	0.66	0.69
MMM	0.94	0.54	0.75	0.74	0.67	0.88	0.75	0.78
MRK	0.84	0.36	0.58	0.60	0.63	0.74	0.64	0.69
NKE	0.84	0.45	0.62	0.65	0.64	0.74	0.65	0.69
PFE	0.72	0.32	0.47	0.52	0.62	0.64	0.58	0.63
PG	0.85	0.30	0.56	0.57	0.60	0.69	0.61	0.65
TRV	0.87	0.44	0.64	0.65	0.64	0.77	0.67	0.71
UNH	0.87	0.34	0.60	0.61	0.61	0.74	0.63	0.67
UTX	0.80	0.31	0.54	0.56	0.57	0.62	0.57	0.60
VZ	0.88	0.38	0.62	0.63	0.64	0.79	0.67	0.71
WMT	0.87	0.33	0.58	0.60	0.63	0.75	0.64	0.69
XOM	0.94	0.31	0.68	0.62	0.61	0.83	0.71	0.72
Total	**0.84**	**0.33**	**0.69**	**0.60**	**0.57**	**065**	**0.58**	**0.62**
Nasdaq-listed stocks								
AAPL	0.64	0.08	0.42	0.19	0.36	0.30	0.32	0.33
CSCO	0.60	0.22	0.34	0.27	0.41	0.31	0.48	0.38
INTC	0.60	0.15	0.34	0.22	0.38	0.28	0.42	0.34
MSFT	0.60	0.16	0.33	0.22	0.38	0.28	0.44	0.35
Total	**0.61**	**0.15**	**0.36**	**0.23**	**0.38**	**0.29**	**0.41**	**0.35**

Table 5: *Correlation of the contribution to price discovery (ISR) with the NYSE share in different transactions size, and with the liquidity (The number of trades per day).*

	NYSE-Listed	NASDAQ-Listed
small trade	0.33	0.30
medium trade	0.10	0.19
big trade	0.27	-0.04
Liquidity	0.33	0.27

Table 6: Macroeconomic announcements days on the period 2011 March 1st to May 31st

Macroeconomics Announcement	Source	Release dates
GDP (Advance, preliminary, final) estimate	BEA	March 25, April 28, May 26
Personal Income, Personal Consumption Expenditures	BEA	March 28, April 29, May 27
International Trade Balance in Goods and Services	BEA	March 10, April 12, May 11
Nonfarm Payroll Employment	BLS	March 4, April 21, May 6
Producer Price Index PPI	BLS	March 16, April 14, May 12
Consumer Price Index CPI	BLS	March 17, April 15, May 13
Industrial Production, Capacity Utilization	FRB	May 17, April 15, March 17
Consumer Credit	FRB	March 7, April 7, May 6
Federal Funds Rate	FRB	March 15, April 27

are wide, the robust IS provides very close bounds that facilitate the interpretation.

5.3 Macroeconomics announcements days

The release of Macroeconomic indicators constitute some of the times where fundamental information arrive in the markets. Interesting insights could thus be investigated by looking at how the different markets behave the days of major macroeconomic news compared to normal days. For this, I identify a set of events from the literature (Andersen et al., 2003; Frijns et al., 2015) and the corresponding dates at which they are released in the sample. I mention that almost all the announcements here happen at 8:30 AM, which is before the markets open. An important comparison of the markets could be done for the news that are released during the trading session, to see for example which market reacts quickly. However, this would required a very long sample as they are typically published only one day per month. Table 6 presents the macroeconomics indicators that I consider and the announcement dates in the sample.

We compute the measures for the announcement days and for the non-announcements days, I obtain that on average NASDAQ's contribution to information share is slightly bigger the days where there is a news (from 0.41 to 0.42). The contribution of NYSE is still greater than for the

Table 7: Markets' contribution to Price discovery on the days of Macroeconomic announcements

	NYSE				NASDAQ			
	IS_u	IS_l	IS	PT	IS_u	IS_l	IS	PT
Announcement	0.84	0.32	0.58	0.57	0.68	0.16	0.42	0.43
Non Announcements	0.84	0.33	0.59	0.58	0.67	0.16	0.41	0.42

NASDAQ but slightly decreases compared to non-announcement days. Since the changes in the numbers are to small it is difficult to convince of a particularity for these news days. If we believe the contribution of NASDAQ has significantly increased, it is difficult to explain why but a reason might be found in the liquidity of NASDAQ. Traders wanting to exploit quickly those public pre-scheduled news, could prefer to do so on the most liquid market. More details, per asset, on which market increases its contribution to price discovery can be found in table B.3.

Conclusion

Among the assets traded on markets places, some are strongly related by arbitrage relationships. This is the case of securities and their derivatives, and assets listed simultaneously in many countries. To determine in which market the efficient price is determined, some measures of price discovery were proven useful in the literature. In this paper, I started by studying the behavior of the popular prices discovery metrics in their relationship with sampling frequency and market microstructure noises. I showed analytically, in some standard microstructure models, that the Information Share measure (IS) of Hasbrouck (1995) and the Permanent-Transitory component measure (PT) of Harris et al. (2002b) are driven by non-informative noises when the sampling interval is small. The IS is identified only between bounds and, when the frequency is low, the bounds are too wide to provide straight conclusions. When the frequency is particularly high, the IS bounds tighten and converge to a unique value which is the same as the PT. But this value is dominated by noises and is not affected by the informative innovation. Using data of NYSE TAQ database, I examined if my conclusions are in line with the data. I observed that indeed the data seem to present the patterns I highlighted. The frequency of the transaction prices might not be high enough to show certain features, but the analysis with mid-quotes of Microsoft confirms my theoretical conclusions.

The price discovery measures are typically used to decide which price is close to the fundamental price. The "closeness" involves two interesting dimensions, the "speed" at which a market incorporates news and the "noise-avoidance" in the mechanism. The two dimensions are economically relevant but confusions come from that a market is not necessarily the best in both dimensions.

A market can be the fastest and the noisiest. At lower frequency the measures capture a mix of the two aspects, while at high frequency, my results showed that they rather capture the avoidance of noise. This is a serious problem because first, many papers use and think of price discovery as the rapidity to process new information. Second, the measures are used in regression to investigate the determinant of a market's efficiency. Those papers conclude for example that market with relative small bid-ask spread are dominating price discovery process. If price discovery were to measure only how markets avoid noises, this conclusion amounts to stating that "Noise is small in this market because noise is small". I then presented new measures of price discovery, that I named Robut IS (ISR) and Robust PT (PTR), that disentangle the two dimensions and clarify the interpretation. They are good at detecting "which market incorporates quickly new information". My overall contribution constitutes many steps forward into the debate on what the price discovery measures are actually capturing.

In the application, I investigated the relative contribution of NYSE and NASDAQ to the price formation of Dow Jones assets. The robust measures seem to improve a little bit on the IS and PT. I found that NYSE captures the big part of volume traded, but NASDAQ is the most liquid with a high level of activity. This implies that NASDAQ mostly runs the orders of small quantities while NYSE runs big quantities orders. In terms of contribution to price discovery for the assets under investigation, NYSE is generally dominant. The contribution of a market appears to be positively correlated with its liquidity. I also computed the correlation between market's contribution and markets share in each category of trade size. It reveals that the contribution of a market is correlated with its share in small size transactions. For NASDAQ listed stocks, there is no correlation with market's share in big size transactions, so large quantities trades do not convey information. I analyze the performance of the measures the days with major macroeconomic announcements and the contribution of NASDAQ to price discovery increases only slightly the days with news. As the announcements considered here are typically done when the market is closed, it is not possible to conclude about the behavior of price discovery measures around the news. There are more insights to investigate with a good database of news released during trading session.

References

Andersen, T., T. Bollerslev, F. Diebold, and P. Labys (2000). Great realizations. *RISK 13*, 105–108. 3, 4, 20

Andersen, T. G., T. Bollerslev, F. X. Diebold, and C. Vega (2003). Micro effects of macro announcements: Real-time price discovery in foreign exchange. *The American Economic Review 93*(1), 38–62. 28

Baillie, R. T., G. G. Booth, Y. Tse, and T. Zabotina (2002). Price discovery and common factor models. *Journal of Financial Markets 5*(3), 309 – 321. 2, 9

Barclay, M. J., T. Hendershott, and D. T. M. N (2003). Competition among trading venues: Information and trading on electronic communications networks. *Journal of Finance 58*, 2637–2665. 4

Chakravarty, S., H. Gulen, and S. Mayhew (2004). Informed trading in stock and option markets. *The Journal of Finance 59*(3), 1235–1257. 4, 7

De Jong, F. (2002). Measures of contributions to price discovery: a comparison. *Journal of Financial Markets 5*(3), 323 – 327. 2, 9

Eun, C. S. and S. Sabherwal (2003). Cross-border listings and price discovery: Evidence from u.s.-listed canadian stocks. *The Journal of Finance 58*(2), 549–575. 2, 4, 9

Frijns, B., I. Indriawan, and A. Tourani-Rad (2015). Macroeconomic news announcements and price discovery: Evidence from canadian-u.s. cross-listed firms. *Journal of Empirical Finance 32*, 35–48. 28

Garbade, K. D. and W. L. Silber (1983). Price movements and price discovery in futures and cash markets. *The Review of Economics and Statistics 65*(2), 289–297. 2

Gonzalo, J. and C. Granger (1995). Estimation of common long-memory components in cointegrated systems. *Journal of Business & Economic Statistics 13*(1), 27–35. 2, 8, 9

Grammig, J. and F. J. Peter (2013). Telltale tails: A new approach to estimating unique market information shares. *Journal of Financial and Quantitative Analysis 48*, 459–488. 8

Harris, F. H., T. H. McInish, and R. A. Wood (2002a). Common factor components versus information shares: a reply. *Journal of Financial Markets 5*(3), 341 – 348. 8, 9

Harris, F. H., T. H. McInish, and R. A. Wood (2002b). Security price adjustment across exchanges: an investigation of common factor components for dow stocks. *Journal of Financial Markets 5*(3), 277 – 308. 2, 9, 29

Hasbrouck, J. (1995). One security, many markets: Determining the contributions to price discovery. *The Journal of Finance 50*(4), 1175–1199. 2, 3, 4, 6, 7, 8, 9, 26, 29

Hasbrouck, J. (2002). Stalking the efficient price in market microstructure specifications: an overview. *Journal of Financial Markets 5*(3), 329 – 339. 2, 9, 15

Huang, R. D. (2002). The quality of ecn and nasdaq market maker quotes. *The Journal of Finance 57*(3), 1285–1319. 4

Jacod, J., Y. Li, P. A. Mykland, M. Podolskij, and M. Vetter (2009). Microstructure noise in the continuous case: The pre-averaging approach. *Stochastic Processes and their Applications 119*(7), 2249 – 2276. 3, 20

Johansen, S. (1991). Estimation and hypothesis testing of cointegration vectors in gaussian vector autoregressive models. *Econometrica 59*(6), 1551–1580. 5

Lanne, M. and H. Lütkepohl (2010). Structural vector autoregressions with nonnormal residuals. *Journal of Business & Economic Statistics 28*(1), 159–168. 8

Lehmann, B. N. (2002). Some desiderata for the measurement of price discovery across markets. *Journal of Financial Markets 5*(3), 259 – 276. 2, 9

Lien, D. and K. Shrestha (2014). Price discovery in interrelated markets. *Journal of Futures Markets 34*(3), 203–219. 8

Menkveld, A. J., S. J. Koopman, and A. Lucas (2007). Modeling around-the-clock price discovery for cross-listed stocks using state space methods. *Journal of Business & Economic Statistics 25*(2), 213–225. 7

Putniņš, T. J. (2013). What do price discovery metrics really measure? *Journal of Empirical Finance 23*, 68 – 83. 3, 7

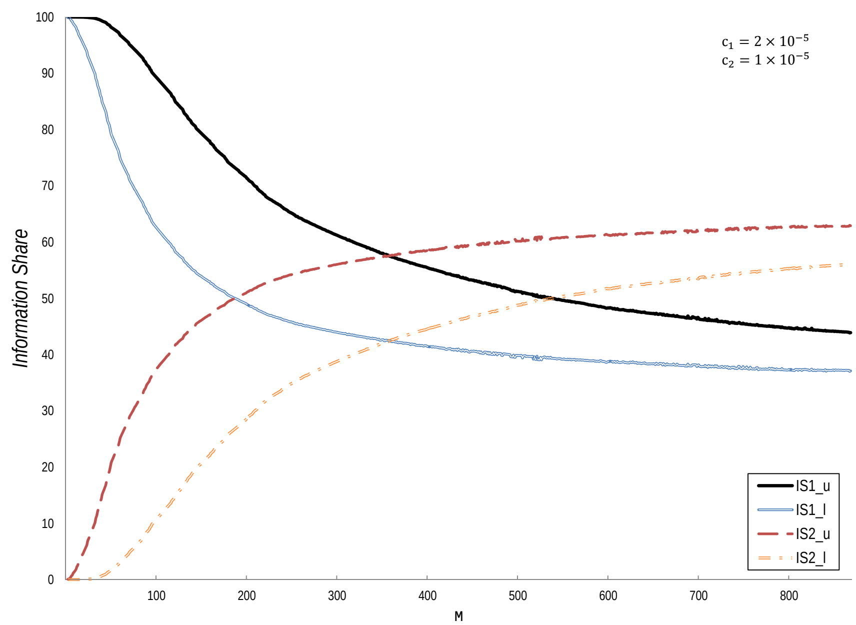
Rigobon, R. (2003). Identification through heteroskedasticity. *The Review of Economics and Statistics 85*(4), 777–792. 8

Yan, B. and E. Zivot (2010). A structural analysis of price discovery measures. *Journal of Financial Markets 13*(1), 1 – 19. 2, 3, 19

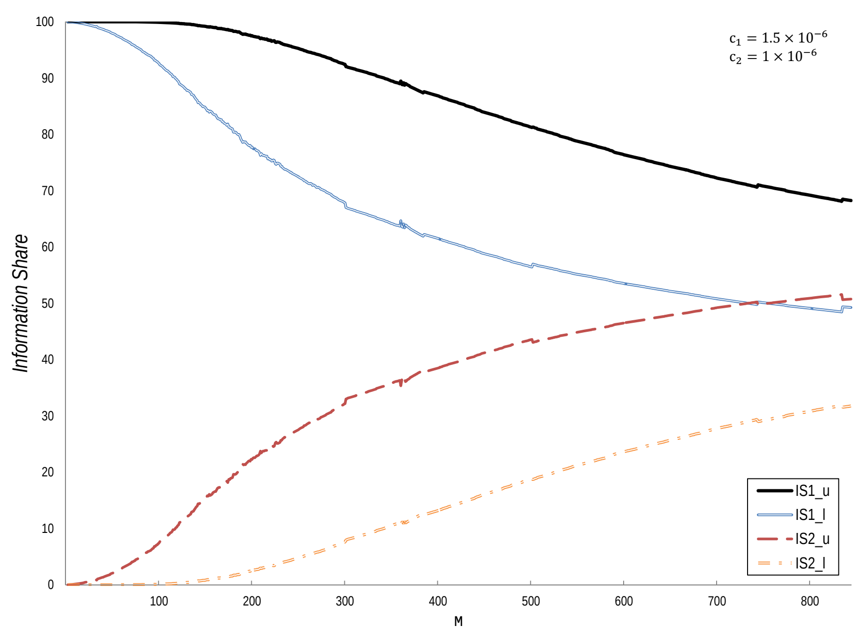
Zhang, F. (2010). High-frequency trading, stock volatility, and price discovery. *Yale University Working Papers*. 3, 20

A Figures

Figure A.1: Model III: IS performance with noise and frequency



(a) Small noises variance

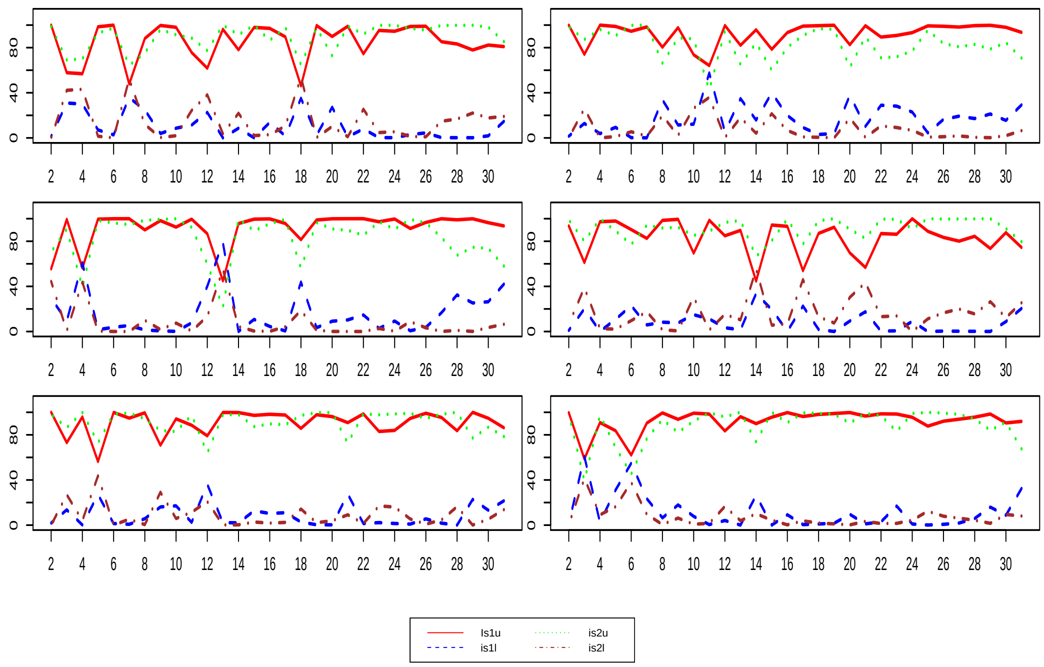


(b) mall difference between the noises

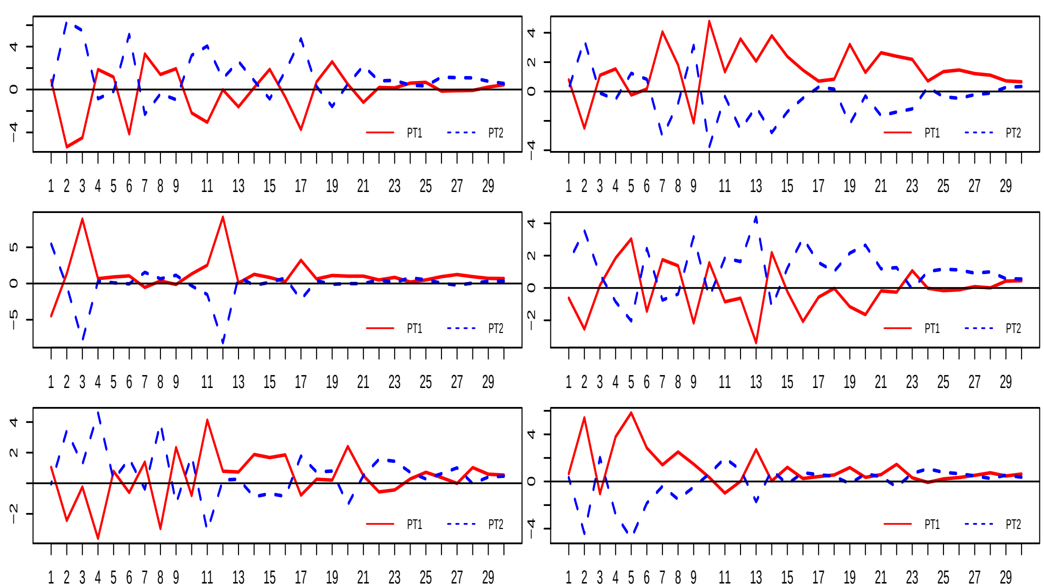
Note: The figure plots the IS model III. The horizontal axis represents $M = 1/h$. The PT (not plotted here) has the same pattern

Figure A.2: IS and PT measures by sampling frequency for American Express

(a) IS bounds AXP



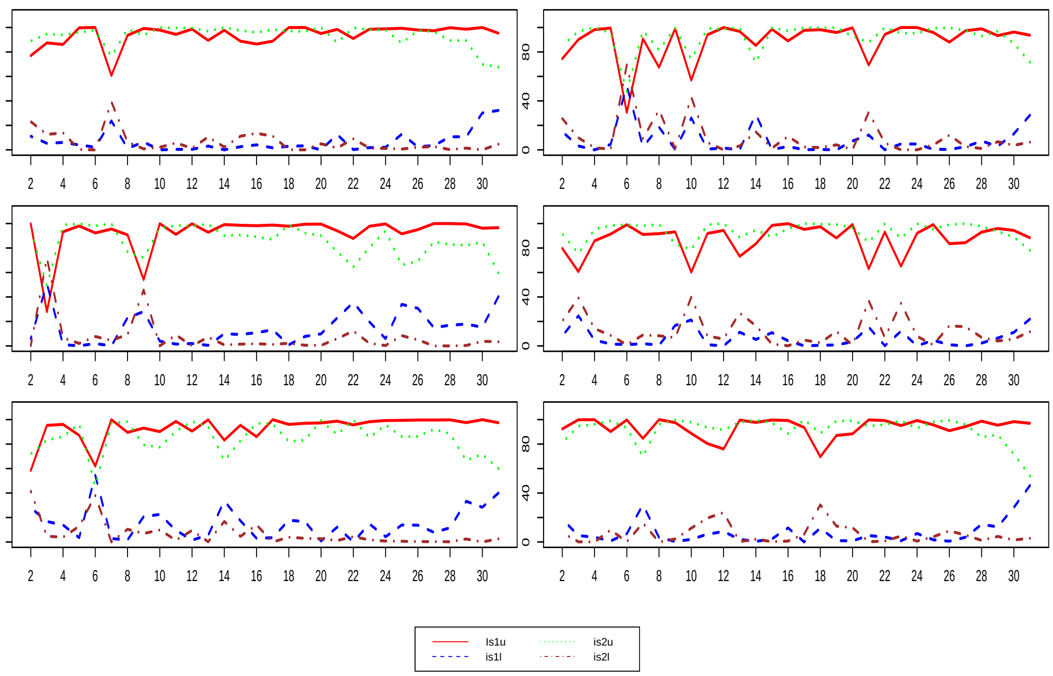
(b) PT for AXP



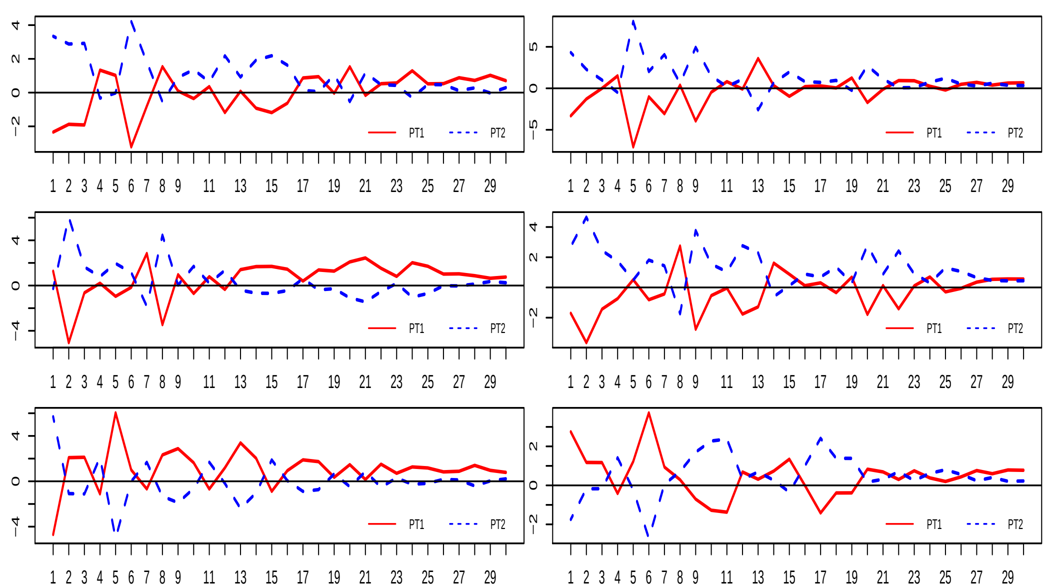
Note: The figures plot 6 chosen day in the database. For each day the data are sampled at different frequency and the measures at computed in a VECM with lag selected by AIC. The horizontal axis represents the sampling frequency M.

Figure A.3: IS and PT measures by sampling frequency for Exxon Mobil Corporation (XOM)

(a) IS bounds for XOM

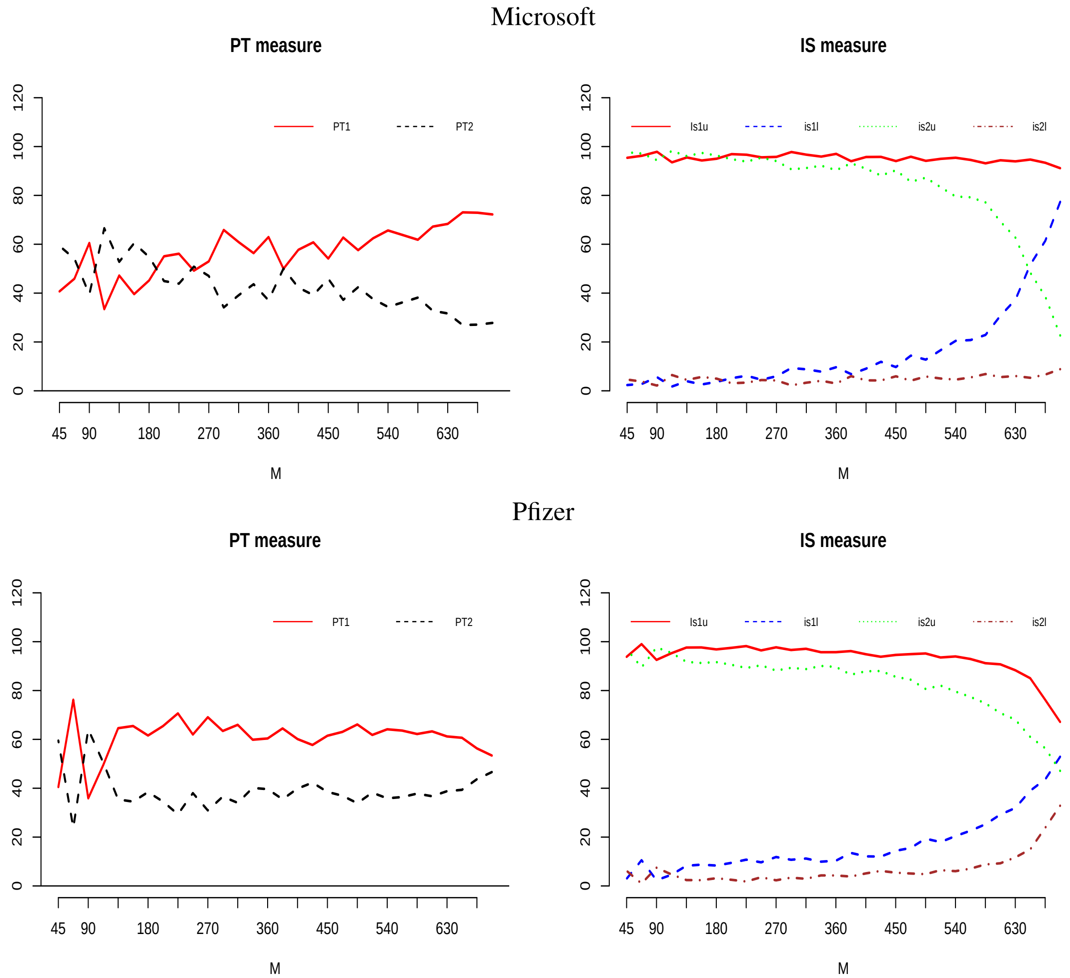


(b) PT for XOM



Note: The figures plot 6 chosen day in the database. For each day the data are sampled at different frequency and the measures at computed in a VECM with lag selected by AIC. The horizontal axis represents the sampling sampling frequency M. e.g: $M = 1$ means 1 observation per 200 s , $M = 30$ means 3 obs. per 20s

Figure A.4: IS and PT measures by sampling frequency



Note: IS and PT for Microsoft and Pfizer Inc on 02/12/2013. The mid-quotes are sampled at different frequency and the measures at computed in a VECM with lag selected by AIC. The horizontal axis represents the sampling frequency M. E.g $M = 1$ *means 1 observation per 200 s ,* $M = 600$ *means 3 obs. per second*

B Tables

Table B.1: *Average daily number and volume of transactions by markets and assets*

	Volume		Liquidity	
Stocks	NYSE	NASDAQ	NYSE	NASDAQ
AXP	1,390,154	1,333,287	5,675	9,175
BA	761,827	939,104	4,265	7,518
CAT	1,509,825	1,258,157	8,587	9,909
CVX	1,638,540	1,387,986	9,221	11,646
DD	1,197,841	932,970	5,618	7,544
DIS	2,205,886	1,279,417	7,123	8,919
GE	7,303,315	7,452,367	8,162	18,419
GS	852,376	957,759	5,407	7,373
HD	1,885,715	1,489,919	5,804	9,231
IBM	1,065,407	979,830	6,294	7,523
JNJ	2,606,299	1,614,322	8,618	10,354
JPM	5,036,588	4,866,551	**11,438**	**22,572**
KO	1,833,110	1,282,143	6,942	9,534
MCD	1,217,757	835,071	5,398	6,390
MMM	877,629	506,236	5,389	4,154
MRK	2,028,357	2,106,988	4,626	9,580
NKE	808,253	479,999	4,164	3,713
PFE	6,601,470	7,418,801	**7,080**	**16,153**
PG	2,010,251	1,732,199	5,725	10,131
TRV	803,587	478,962	3,893	3,877
UNH	1,331,712	974,829	5,753	7,084
UTX	852,967	766,713	4,778	6,367
VZ	2,261,393	2,160,213	5,614	10,639
WMT	2,136,040	1,521,625	6,743	9,916
XOM	4,649,096	2,546,214	15,937	17,663
	NYSE Arca	NASDAQ	NYSE Arca	NASDAQ
AAPL	2,318,123	3,944,326	17,123	27,099
CSCO	7,573,610	13,954,385	14,945	22,991
INTC	6,455,527	16,635,294	15,125	30,653
MSFT	5,513,734	14,580,294	14,825	27,540

Note: The period is from the 01/03 to 30/05/2011. liquidity=number of transactions per day; volume=volume of trades per day.

Table B.2: *Averages daily share of each market in transactions, by category of transactions size,*

	Small size		**Medium size**		**Big size**	
Stock	**NYSE**	**NASDAQ**	**NYSE**	**NASDAQ**	**NYSE**	**NASDAQ**
AXP	0.32	0.68	0.51	0.49	0.71	0.29
BA	0.32	0.68	0.55	0.45	0.65	0.35
CAT	0.41	0.59	0.66	0.34	0.72	0.28
CVX	0.37	0.63	0.69	0.31	0.79	0.21
DD	0.35	0.65	0.66	0.34	0.81	0.19
DIS	0.39	0.61	0.67	0.33	0.86	0.14
GE	0.29	0.71	0.28	0.72	0.57	0.43
GS	0.40	0.60	0.60	0.40	0.58	0.42
HD	0.33	0.67	0.55	0.45	0.71	0.29
IBM	0.41	0.59	0.62	0.38	0.69	0.31
JNJ	0.41	0.59	0.59	0.41	0.78	0.22
JPM	0.30	0.70	0.38	0.62	0.66	0.34
KO	0.33	0.67	0.68	0.32	0.79	0.21
MCD	0.38	0.62	0.67	0.33	0.79	0.21
MMM	0.52	0.48	0.75	0.25	0.81	0.19
MRK	0.28	0.72	0.36	0.64	0.56	0.44
NKE	0.47	0.53	0.72	0.28	0.80	0.20
PFE	0.30	0.70	0.27	0.73	0.48	0.52
PG	0.32	0.68	0.61	0.39	0.75	0.25
TRV	0.44	0.56	0.77	0.23	0.85	0.15
UNH	0.38	0.62	0.63	0.37	0.75	0.25
UTX	0.37	0.63	0.68	0.32	0.76	0.24
VZ	0.31	0.69	0.33	0.67	0.67	0.33
WMT	0.35	0.65	0.57	0.43	0.78	0.22
XOM	0.41	0.59	0.69	0.31	0.83	0.17
Total	0.37	0.63	0.58	0.42	0.73	0.27

Note: *Let DM the average transactions size in a day: Small size* $\equiv quantity < DM$, *medium size:* $\equiv DM \leq quantity \leq 2DM$, *big size:* $\equiv quantity > 2DM$.

Table B.3: NYSE contribution by asset on the days of Macroeconomic announcements

	ISR_u		ISR_l		ISR		PTR	
	A	**N**	**A**	**N**	**A**	**N**	**A**	**N**
AXP	0.85	0.85	0.26	0.28	0.55	0.57	0.54	0.56
BA	0.65	0.63	0.15	0.14	0.40	0.39	0.37	0.35
CAT	0.80	0.78	0.22	0.20	0.51	0.49	0.49	0.47
CVX	0.89	0.89	0.28	0.33	0.59	0.61	0.61	0.63
DD	0.82	0.82	0.27	0.28	0.55	0.55	0.55	0.54
DIS	0.89	0.90	0.32	0.38	0.61	0.64	0.61	0.65
GE	0.83	0.78	0.38	0.36	0.60	0.57	0.57	0.52
GS	0.75	0.71	0.25	0.23	0.50	0.47	0.49	0.46
HD	0.86	0.88	0.29	0.35	0.57	0.62	0.56	0.61
IBM	0.76	0.73	0.26	0.24	0.51	0.49	0.50	0.47
JNJ	0.89	0.91	0.39	0.42	0.64	0.67	0.64	0.67
JPM	0.90	0.88	0.28	0.29	0.59	0.59	0.59	0.59
KO	0.87	0.87	0.27	0.30	0.57	0.58	0.57	0.58
MCD	0.87	0.87	0.40	0.40	0.63	0.63	0.64	0.62
MMM	0.93	0.94	0.51	0.55	0.72	0.75	0.72	0.77
MRK	0.85	0.84	0.33	0.38	0.59	0.61	0.57	0.58
NKE	0.85	0.84	0.48	0.44	0.66	0.64	0.64	0.62
PFE	0.75	0.71	0.31	0.33	0.53	0.52	0.49	0.46
PG	0.84	0.85	0.27	0.31	0.55	0.58	0.54	0.57
TRV	0.85	0.88	0.40	0.46	0.63	0.67	0.62	0.66
UNH	0.87	0.87	0.34	0.34	0.60	0.61	0.60	0.60
UTX	0.80	0.81	0.30	0.31	0.55	0.56	0.53	0.55
VZ	0.86	0.89	0.34	0.41	0.60	0.65	0.59	0.64
WMT	0.87	0.86	0.33	0.33	0.60	0.60	0.59	0.58
XOM	0.94	0.94	0.31	0.31	0.62	0.62	0.69	0.68
Total	0.84	0.84	0.32	0.33	0.58	0.59	0.57	0.58

$IS_l = (IS_u + IS_l)/2$,,A=announcement, N=Non announcement

C Analytical formulas of the measures

We present calculations by skipping some details. The detailed calculations can be found (Online here)

C.1 General results

Consider $p_t = \begin{pmatrix} p_{1t} & p_{2t} \end{pmatrix}'$ admitting the VMA(1) : $\Delta p_t = e_t + \Theta e_{t-1}$

$$\text{With } \Omega = var\left(\varepsilon_t\right) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \text{ and } \Psi\left(1\right) = I + \Theta = \begin{pmatrix} c & 1+d \\ c & 1+d \end{pmatrix}.$$

The goal is to solve for Θ and Ω given the structural parameters. Let

$$C_0 \equiv var(\Delta p_t) = \begin{pmatrix} v_1^2 & v_{12} \\ v_{12} & v_2^2 \end{pmatrix} \text{ and } C_1 \equiv cov(\Delta p_t, \Delta p_{t-h}^{'}) = \begin{pmatrix} m_1 & m_{12} \\ m_{21} & m_2 \end{pmatrix} \tag{27}$$

Using the VMA(1) 8 gives

$$\begin{aligned} C_0 &= \Omega + \Theta\Omega\Theta' && (28) \\ C_1 &= \Theta\Omega && (29) \end{aligned}$$

By multiplying 28 by Θ and using 29 then

$$C_1 - \Theta C_0 + \Theta C_1 \Theta' = 0 \tag{30}$$

$$\begin{aligned} \Theta C_1 \Theta' &= \begin{pmatrix} -1+c & 1+d \\ c & d \end{pmatrix} \begin{pmatrix} m_1 & m_{12} \\ m_{21} & m_2 \end{pmatrix} \begin{pmatrix} -1+c & c \\ 1+d & d \end{pmatrix} \\ &= \begin{pmatrix} c^2 m_1 + d^2 m_2 + cd\left(m_{12}+m_{21}\right) + c\left(-2m_1 + m_{12} + m_{21}\right) + d\left(2m_2 - m_{12} - m_{21}\right) \\ +m_1 + m_2 - m_{12} - m_{21} \\ c^2 m_1 + d^2 m_2 + cd\left(m_{12}+m_{21}\right) + c\left(-m_1 + m_{21}\right) + d\left(m_2 - m_{12}\right) \\ c^2 m_1 + d^2 m_2 + cd\left(m_{12}+m_{21}\right) + c\left(-m_1 + m_{12}\right) + d\left(-m_{21} + m_2\right) \\ c^2 m_1 + d^2 m_2 + cd\left(m_{12}+m_{21}\right) \end{pmatrix} \end{aligned}$$

Set $Q = c^2 m_1 + d^2 m_2 + cd\left(m_{12} + m_{21}\right)$ then

$$\Theta C_1 \Theta' = \begin{pmatrix} Q + c\left(-2m_1 + m_{12} + m_{21}\right) + d\left(2m_2 - m_{12} - m_{21}\right) + m_1 + m_2 - m_{12} - m_{21} \\ Q + c\left(-m_1 + m_{21}\right) + d\left(m_2 - m_{12}\right) \\ Q + c\left(-m_1 + m_{12}\right) + d\left(m_2 - m_{21}\right) \\ Q \end{pmatrix}$$

We stack the lines to ease the presentation ΘC_0.

Using equation 30

$$\begin{aligned} 0 &= \Theta C_1 \Theta' + C_1 - \Theta C_0 \\ &= \begin{pmatrix} Q + c\left(-2m_1 + m_{12} + m_{21} - v_1^2\right) + d\left(2m_2 - m_{12} - m_{21} - v_{12}\right) \\ \quad +2m_1 + m_2 - m_{12} - m_{21} + v_1^2 - v_{12} \\ Q + c\left(-m_1 + m_{21} - v_{12}\right) + d\left(m_2 - m_{12} - v_2^2\right) + m_{12} + v_{12} - v_2^2 \\ Q + c\left(-m_1 + m_{12} - v_1^2\right) + d\left(m_2 - m_{21} - v_{12}\right) + m_{21} \\ Q - cv_{12} - dv_2^2 + m_2 \end{pmatrix} \end{aligned}$$

Subtracting the 2nd from the 3rd line

$$d\left(m_{12} - m_{21} + v_2^2 - v_{12}\right) = c\left(-m_{12} + m_{21} + v_1^2 - v_{12}\right) + m_{12} - m_{21} - v_2^2 + v_{12} \tag{31}$$

$$\begin{aligned} d &= c\frac{\left(-m_{12} + m_{21} + v_1^2 - v_{12}\right)}{m_{12} - m_{21} + v_2^2 - v_{12}} + \frac{m_{12} - m_{21} - v_2^2 + v_{12}}{m_{12} - m_{21} + v_2^2 - v_{12}} \\ &= cF + G \end{aligned} \tag{32}$$

with

$$F = \frac{-m_{12} + m_{21} + v_1^2 - v_{12}}{m_{12} - m_{21} + v_2^2 - v_{12}} \text{ and } G = \frac{m_{12} - m_{21} - v_2^2 + v_{12}}{m_{12} - m_{21} + v_2^2 - v_{12}} \tag{33}$$

Which is plugged into the quadratic equation (4th line):

$$c^2 m_1 + d^2 m_2 + cd\left(m_{12} + m_{21}\right) - cv_{12} - dv_2^2 + m_2 = 0 \tag{34}$$

$$
\begin{aligned}
0 &= c^2 m_1 + d^2 m_2 + cd\left(m_{12}+m_{21}\right) - c v_{12} - d v_2^2 + m_2 \\
&= c^2 m_1 + (cF+G)^2 m_2 + c\,(cF+G)\left(m_{12}+m_{21}\right) - c v_{12} - (cF+G)\, v_2^2 + m_2 \\
&= c^2\left(m_1 + F^2 m_2 + F\left(m_{12}+m_{21}\right)\right) + c\left[2FGm_2 + G\left(m_{12}+m_{21}\right) - v_{12} - F v_2^2\right] \\
&\quad + G^2 m_2 - G v_2^2 + m_2
\end{aligned}
$$

$$
c^2 + c\frac{\left[2FGm_2 + G\left(m_{12}+m_{21}\right) - v_{12} - F v_2^2\right]}{m_1 + F^2 m_2 + F\left(m_{12}+m_{21}\right)} + \frac{G^2 m_2 - G v_2^2 + m_2}{\left(m_1 + F^2 m_2 + F\left(m_{12}+m_{21}\right)\right)}
$$

$$
\Delta = \left[\frac{2FGm_2 + G\left(m_{12}+m_{21}\right) - v_{12} - F v_2^2}{m_1 + F^2 m_2 + F\left(m_{12}+m_{21}\right)}\right]^2 - 4\frac{G^2 m_2 - G v_2^2 + m_2}{\left(m_1 + F^2 m_2 + F\left(m_{12}+m_{21}\right)\right)}
$$

$$
\boxed{
\begin{aligned}
c &= -\frac{1}{2}\frac{2FGm_2 + G\left(m_{12}+m_{21}\right) - v_{12} - F v_2^2}{m_1 + F^2 m_2 + F\left(m_{12}+m_{21}\right)} \pm \frac{1}{2}\sqrt{\Delta} \\
d &= cF + g
\end{aligned}
\qquad (35)}
$$

then

$$
\Omega = \Theta^{-1} C_1 = -\begin{pmatrix} -1+c & d \\ c & -1+d \end{pmatrix}^{-1} \begin{pmatrix} m_1 & m_{12} \\ m_{21} & m_2 \end{pmatrix} \tag{36}
$$

The PT measure We have computed $\psi \equiv \begin{pmatrix} \psi_{11} & \psi_{12} \end{pmatrix} = \begin{pmatrix} c & 1+d \end{pmatrix}$

$$
PT_1 = \frac{c}{1+c+d} \text{ and } PT_2 = \frac{1+d}{1+c+d} \tag{37}
$$

The information share bounds The total IS is

$$
\begin{aligned}
\psi\Omega\psi' &= \begin{pmatrix} c & 1+d \end{pmatrix} \Omega \begin{pmatrix} c \\ 1+d \end{pmatrix} \\
&= \left[c^2 \sigma_{11} + 2\sigma_{12} c\,(1+d) + (1+d)^2 \sigma_{22}\right]
\end{aligned} \tag{38}
$$

The IS bounds for market 1 are

$$\begin{aligned} IS1u &= \left(\psi_{11}\sqrt{\sigma_{11}}+\psi_{12}\rho\sqrt{\sigma_{22}}\right)^2/\psi\Omega\psi \\ &= \left(c\sqrt{\sigma_{11}}+(1+d)\,\sigma_{12}\left(\sqrt{\sigma_{11}}\right)^{-1}\right)^2/\psi\Omega\psi \\ IS1l &= \left(\psi_{11}\sqrt{\sigma_{11}}\sqrt{(1-\rho^2)}\right)^2/\psi\Omega\psi \\ &= c^2\sigma_{11}\left(1-\sigma_{12}^2/\sigma_{11}\sigma_{22}\right)/\psi\Omega\psi \end{aligned} \tag{39}$$

and for market 2

$$\begin{aligned} IS2u &= \left(\psi_{12}\sqrt{\sigma_{22}}+\psi_{11}\rho\sqrt{\sigma_{11}}\right)^2/\psi\Omega\psi \\ &= \left((1+d)\sqrt{\sigma_{22}}+c\sigma_{12}\left(\sqrt{\sigma_{22}}\right)^{-1}\right)^2/\psi\Omega\psi \\ IS2l &= \left(\psi_{12}\sqrt{\sigma_{22}}\sqrt{(1-\rho^2)}\right)^2/\psi\Omega\psi \\ &= (1+d)^2\sigma_{22}\left(1-\sigma_{12}^2/\sigma_{11}\sigma_{22}\right)/\psi\Omega\psi \end{aligned} \tag{40}$$

C.2 Model I: A two-market "Roll" model.

Here $m_{th}=m_{th-h}+\eta_{th}$, the is innovation $\eta_{th}=\sigma_h\mathcal{N}(0,1)$ and $\sigma(h)$converges to zero with$h$

$$\begin{aligned} p_{1th} =& m_{th}+c_1\varepsilon_{1th} \\ p_{2th} =& m_{th}+c_2\varepsilon_{2th} \end{aligned} \tag{41}$$

With $\varepsilon_{it}\sim\mathcal{N}(0,1)$, $E(\eta_{th}\varepsilon_{it})=0$, i=1,2. $c_1,c_2>0$

Equation 33 gives $G=-1$, $F=c_1^2c_2^{-2}$, thus $1+d=cc_1^2c_2^{-2}$

In the 2nd degree equation 34

$$\Delta = c_1^{-4}\sigma_h^4\left[\sigma_h^2+4\left(c_1^{-2}+c_2^{-2}\right)^{-1}\right]$$

$$c = -\frac{1}{2}c_1^{-2}\sigma_h^2+\frac{1}{2}c_1^{-2}\sigma_h\sqrt{\sigma_h^2+4\left(c_1^{-2}+c_2^{-2}\right)^{-1}}$$

set $\kappa=-\frac{1}{2}\sigma_h^2+\frac{\sigma_h}{2}\sqrt{\sigma_h^2+4\left(c_1^{-2}+c_2^{-2}\right)^{-1}}$ then

$$c = c_1^{-2}\kappa \text{ and } 1+d = c_2^{-2}\kappa$$

using 36 , with K=$\left[1 - c_1^{-2}\kappa - c_2^{-2}\kappa\right]^{-1}$, $\Omega \quad = \quad K\begin{pmatrix} c_1^2\left(1-c_2^{-2}\kappa\right) & \kappa \\ \kappa & c_2^2\left(1-c_1^{-2}\kappa\right) \end{pmatrix}$

The PT measure we have $\begin{pmatrix} c & 1+d \end{pmatrix} = \begin{pmatrix} c_1^{-2}\kappa & c_2^{-2}\kappa \end{pmatrix}$ so

$$PT_1 = \frac{c_1^{-2}}{c_1^{-2}+c_2^{-2}} \text{ and } PT_2 = \frac{c_2^{-2}}{c_1^{-2}+c_2^{-2}}$$

The IS bounds The total IS 38 is here $\psi\Omega\psi' = K\kappa^2\left(c_1^{-2}+c_2^{-2}\right)$. Using 39 and 40:

The bounds for market 1 are

- $IS1u = \frac{c_1^{-2}}{\left(c_1^{-2}+c_2^{-2}\right)\left(1-c_2^{-2}\kappa\right)}$ and $IS1l = \frac{c_1^{-2}K^{-1}}{\left(c_1^{-2}+c_2^{-2}\right)\left(1-c_1^{-2}\kappa\right)}$

And for market 2

- $IS2u = \frac{c_2^{-2}}{\left(c_1^{-2}+c_2^{-2}\right)\left(1-c_1^{-2}\kappa\right)}$ and $IS2l = \frac{c_2^{-2}K^{-1}}{\left(c_1^{-2}+c_2^{-2}\right)\left(1-c_2^{-2}\kappa\right)}$

The Lemma 1 and the Propositions 1,2,3 are proven.

C.3 Model II: The Roll model with a delayed market

The prices system is

$$\begin{aligned} m_{th} &= m_{th-h} + \eta_{th} \\ p_{1th} &= m_{th} + c_1\varepsilon_{1th} \\ p_{2th} &= m_{th-\delta} + c_2\varepsilon_{2th} \end{aligned}$$

The second market is delayed of δ.

To specify the how h moves with respect to δ, we set $\delta = b \times l$, $h = k \times l$. l is a short time pace and there is a white noise u_l, with $var(\mu_l) = \sigma^2$ and

$$
\begin{aligned}
m_{tkl} &= m_{tkl-l} + u_{tkl} \\
&= m_{tkl-2l} + u_{tkl} + u_{tkl-l} \\
&\vdots \\
m_{tkl} &= m_{tkl-kl} + \sum_{j=0}^{k-1} u_{tkl-jl}
\end{aligned}
$$

Thus $m_{th} = m_{th-h} + \eta_{th}$ with $\eta_{th} = \sum_{j=0}^{k-1} u_{tkl-jl}$ and $var(\eta_{th}) = h\sigma^2$.

$$
\begin{aligned}
\Delta p_{1th} = \quad & \Delta m_{th} + c_1 \Delta\varepsilon_{1th} &= \quad & \textstyle\sum_{j=0}^{k-1} u_{tkl-jl} + c_1 \Delta\varepsilon_{1th} \\
\Delta p_{2th} = \quad & \Delta m_{th-\delta} + c_2 \Delta\varepsilon_{2th} &= \quad & \textstyle\sum_{j=b}^{k+b-1} u_{tkl-jl} + c_2 \Delta\varepsilon_{2th}
\end{aligned}
$$

Then we easily compute

$$
C_0 = \begin{pmatrix} h\sigma^2 + 2c_1^2 & (h-\delta)\sigma^2 \\ (h-\delta)\sigma^2 & h\sigma^2 + 2c_2^2 \end{pmatrix} \text{ and } C_1 = \begin{pmatrix} -c_1^2 & 0 \\ \delta\sigma^2 & -c_2^2 \end{pmatrix}
$$

We compute F,G from 33

$$
F = \frac{-\delta\sigma^2 + h\sigma^2 + 2c_1^2 - (h-\delta)\sigma^2}{\delta\sigma^2 + h\sigma^2 + 2c_2^2 - (h-\delta)\sigma^2} = \frac{c_1^2}{\delta\sigma^2 + c_2^2}
$$

$$
G = \frac{\delta\sigma^2 - h\sigma^2 - 2c_2^2 + (h-\delta)\sigma^2}{\delta\sigma^2 + h\sigma^2 + 2c_2^2 - (h-\delta)\sigma^2} = \frac{-c_2^2}{\delta\sigma^2 + c_2^2}
$$

Thus $d\left(\delta\sigma^2 + c_2^2\right) = cc_1^2 - c_2^2$
Which is plugged into the elements of 35:

$$
\left(\delta\sigma^2 + c_2^2\right)^2 \left(2FGm_2 + Gm_{12} - v_{12} - Fv_2^2\right) = -\sigma^2 \left[\left(\delta\sigma^2 + c_2^2\right)\left(\sigma^2\delta(h-\delta) + h\left(c_1^2 + c_2^2\right)\right) + 2c_1^2 c_2^2 \delta\right]
$$

$$
\left(\delta\sigma^2 + c_2^2\right)^2 \left(G^2 m_2 - Gv_2^2 + m_2\right) = c_2^2 \sigma^2 \left[\sigma^2\delta(h-\delta) + hc_2^2\right]
$$

$$
\left(\delta\sigma^2 + c_2^2\right)^2 \left(m_1 + F^2 m_2 + Fm_{12}\right) = -c_1^2 \left[\left(\delta\sigma^2\right)^2 + \delta\sigma^2 c_2^2 + c_1^2 c_2^2 + c_2^4\right]
$$

$$\Delta = \left[\frac{\sigma^2\left[\left(\delta\sigma^2+c_2^2\right)\left(\sigma^2\delta\left(h-\delta\right)+h\left(c_1^2+c_2^2\right)\right)+2c_1^2c_2^2\delta\right]}{c_1^2\left[\left(\delta\sigma^2\right)^2+\delta\sigma^2c_2^2+c_1^2c_2^2+c_2^4\right]}\right]^2+4\frac{c_2^2\sigma^2\left[\sigma^2\delta\left(h-\delta\right)+hc_2^2\right]}{c_1^2\left[\left(\delta\sigma^2\right)^2+\delta\sigma^2c_2^2+c_1^2c_2^2+c_2^4\right]}$$

$$c = -\frac{1}{2}\frac{\sigma^2\left[\left(\delta\sigma^2+c_2^2\right)\left(\sigma^2\delta\left(h-\delta\right)+h\left(c_1^2+c_2^2\right)\right)+2c_1^2c_2^2\delta\right]}{c_1^2\left[\left(\delta\sigma^2\right)^2+\delta\sigma^2c_2^2+c_1^2c_2^2+c_2^4\right]}\pm\frac{1}{2}\sqrt{\Delta}$$

$$d = \left[-\frac{1}{2}\frac{\sigma^2\left[\left(\delta\sigma^2+c_2^2\right)\left(\sigma^2\delta\left(h-\delta\right)+h\left(c_1^2+c_2^2\right)\right)+2c_1^2c_2^2\delta\right]}{c_1^2\left[\left(\delta\sigma^2\right)^2+\delta\sigma^2c_2^2+c_1^2c_2^2+c_2^4\right]}\pm\frac{1}{2}\sqrt{\Delta}\right]\frac{c_1^2}{\delta\sigma^2+c_2^2}-\frac{c_2^2}{\delta\sigma^2+c_2^2}$$

then using 36

$$\Omega = (1-c-d)^{-1}\begin{pmatrix} -c_1^2(-1+d)-d\delta\sigma^2 & dc_2^2 \\ cc_1^2+(-1+c)\,\delta\sigma^2 & -(-1+c)\,c_2^2 \end{pmatrix}$$

And then the PT and the IS are computed by replacing in 39, 40, 37.

The PT measure

$$PT_1 = \frac{c}{1+c+d} \text{ and } PT_2 = \frac{1+d}{1+c+d} \tag{42}$$

The information share bounds The total IS is

$$\begin{aligned}
\psi\Omega\psi' &= \begin{pmatrix} c & 1+d \end{pmatrix}\Omega\begin{pmatrix} c \\ 1+d \end{pmatrix} \\
&= \left[c^2\sigma_{11}+2\sigma_{12}c\,(1+d)+(1+d)^2\,\sigma_{22}\right] \\
&= c^2\times(-1+c+d)^{-1}\left(c_1^2(-1+d)+d\delta\sigma^2\right)+2dc_2^2(1-c-d)^{-1}c\,(1+d)-(1-c-d)^{-1}(-1+c)\,c_2^2
\end{aligned}$$

The IS bounds for market 1 are

$$IS1u = \frac{\left(c^2(-1+c+d)^{-1}\left(c_1^2(-1+d)+d\delta\sigma^2\right)+\frac{(1+d)^2\left(dc_2^2(1-c-d)^{-1}\right)^2}{\left((-1+c+d)^{-1}\left(c_1^2(-1+d)+d\delta\sigma^2\right)\right)}/+4c(1+d)dc_2^2(1-c-d)^{-1}\right)}{c^2\times(-1+c+d)^{-1}\left(c_1^2(-1+d)+d\delta\sigma^2\right)+2dc_2^2(1-c-d)^{-1}c\,(1+d)-(1-c-d)^{-1}(-1+c)\,c_2^2}$$

$$IS1l = \frac{c^2(-1+c+d)^{-1}\left(c_1^2(-1+d)+d\delta\sigma^2\right)\left(1-\frac{(1+d)^2\left(dc_2^2(1-c-d)^{-1}\right)^2}{\left(c_1^2(-1+d)+d\delta\sigma^2\right)\left((-1+c)c_2^2\right)}\right)}{c^2\times(-1+c+d)^{-1}\left(c_1^2(-1+d)+d\delta\sigma^2\right)+2dc_2^2(1-c-d)^{-1}c\,(1+d)-(1-c-d)^{-1}(-1+c)\,c_2^2}$$

and for market 2

$$IS2u = 1 - \frac{c^2(-1+c+d)^{-1}\left(c_1^2(-1+d)+d\delta\sigma^2\right)\left(1-\frac{(1+d)^2\left(dc_2^2(1-c-d)^{-1}\right)^2}{\left(c_1^2(-1+d)+d\delta\sigma^2\right)\left((-1+c)c_2^2\right)}\right)}{c^2\times(-1+c+d)^{-1}\left(c_1^2(-1+d)+d\delta\sigma^2\right)+2dc_2^2(1-c-d)^{-1}c(1+d)-(1-c-d)^{-1}(-1+c)c_2^2}$$

$$IS2l = 1 - \frac{\left(c^2(-1+c+d)^{-1}\left(c_1^2(-1+d)+d\delta\sigma^2\right)+\frac{(1+d)^2\left(dc_2^2(1-c-d)^{-1}\right)^2}{\left((-1+c+d)^{-1}\left(c_1^2(-1+d)+d\delta\sigma^2\right)\right)}/+4c(1+d)dc_2^2(1-c-d)^{-1}\right)}{c^2\times(-1+c+d)^{-1}\left(c_1^2(-1+d)+d\delta\sigma^2\right)+2dc_2^2(1-c-d)^{-1}c(1+d)-(1-c-d)^{-1}(-1+c)c_2^2}$$

C.4 Model III: two markets with public and private information

We have $\lambda_h, \sigma(h) \overset{h\to 0}{\longrightarrow} 0$ and

$$\begin{aligned} m_t &= m_{t-h} + \lambda_h \eta_{1th} + \eta_{th} \\ p_{1th} &= m_{th} + \eta_{1th} + c_1 \varepsilon_{1th} \\ P_{2th} &= m_{th-h} + c_2 \varepsilon_{2th} \end{aligned} \tag{43}$$

thus

$$C_0 = \begin{pmatrix} (\lambda_h+1)^2+1+\sigma_h^2+2c_1^2 & -\lambda_h \\ -\lambda_h & \lambda_h^2+\sigma_h^2+2c_2^2 \end{pmatrix} \text{ and } C_1 = \begin{pmatrix} -(\lambda_h+1)-c_1^2 & 0 \\ \lambda_h(\lambda_h+1)+\sigma_h^2 & -c_2^2 \end{pmatrix} \tag{44}$$

Using the equation 33

$$F = c_2^{-2}\left[(\lambda_h+1)^2+\sigma_h^2+c_1^2\right] \quad \text{and} \quad G = -c_2^{-2}\left(\sigma_h^2+\lambda_h^2+c_2^2+\lambda_h\right)$$

For $h \simeq 0$ we consider the development at the order of λ_h and σ_h. That is
$F = c_2^{-2}\left(1+c_1^2\right)$, $G = -\left(1+c_2^{-2}\lambda_h\right)$ thus $d = c_2^{-2}\left(1+c_1^2\right)c - \left(1+c_2^{-2}\lambda_h\right)$ and we have

$$C_0 = \begin{pmatrix} 2\lambda_h+2+2c_1^2 & -\lambda_h \\ -\lambda_h & 2c_2^2 \end{pmatrix} \quad \text{and} \quad C_1 = \begin{pmatrix} -1-c_1^2 & 0 \\ \lambda_h & -c_2^2 \end{pmatrix}$$

In equation 34

$$m_1 + F^2 m_2 + F(m_{12}+m_{21}) = -\left(1+c_1^2\right)\left[1+c_2^{-2}\left(1+c_1^2\right)-c_2^{-2}\lambda_h\right]$$

$$2FGm_2+G(m_{12}+m_{21})-v_{12}-Fv_2^2 \quad = \quad \left(c_2^{-2}\lambda_h\right)\left(2\left(1+c_1^2\right)-\lambda_h\right)$$

$$G^2m_2-Gv_2^2+m_2 \quad = \quad -c_2^2\left(2+c_2^{-2}\lambda_h\right)^2$$

$$\begin{aligned}
\Delta &= \left(c_2^{-2}\lambda_h\right)^2\left(2\left(1+c_1^2\right)-\lambda_h\right)^2-4c_2^2\left(2+c_2^{-2}\lambda_h\right)^2\left(1+c_1^2\right)\left[1+c_2^{-2}\left(1+c_1^2\right)-c_2^{-2}\lambda_h\right] \\
&= c_2^{-4}\lambda_h^2\left(4\left(1+c_1^2\right)^2+\lambda_h^2-4\left(1+c_1^2\right)\lambda_h\right)-4\left(4+4c_2^{-2}\lambda_h+c_2^{-4}\lambda_h^2\right)c_2^2\left(1+c_1^2\right)\left[1+c_2^{-2}\left(1+c_1^2\right)\right] \\
&\quad +4\left(4+4c_2^{-2}\lambda_h+c_2^{-4}\lambda_h^2\right)c_2^2\left(1+c_1^2\right)c_2^{-2}\lambda_h
\end{aligned}$$

$$\begin{aligned}
c &= \frac{c_2^{-2}\lambda_h\left(1+c_1^2\right)+\sqrt{\lambda_h\left(1+c_1^2\right)\left(1+c_2^{-2}\left(1+c_1^2\right)\right)}}{\left(1+c_1^2\right)\left[1+c_2^{-2}\left(1+c_1^2\right)-c_2^{-2}\lambda_h\right]} \\
d &= c_2^{-2}\left(1+c_1^2\right)c-\left(1+c_2^{-2}\lambda_h\right) \\
&= c_2^{-2}\left(1+c_1^2\right)\frac{c_2^{-2}\lambda_h\left(1+c_1^2\right)+\sqrt{\lambda_h\left(1+c_1^2\right)\left(1+c_2^{-2}\left(1+c_1^2\right)\right)}}{\left(1+c_1^2\right)\left[1+c_2^{-2}\left(1+c_1^2\right)-c_2^{-2}\lambda_h\right]}-c_2^{-2}\lambda_h-1 \\
1+d &= c_2^{-2}\frac{c_2^{-2}\lambda_h\left(1+c_1^2\right)+\sqrt{\lambda_h\left(1+c_1^2\right)\left(1+c_2^{-2}\left(1+c_1^2\right)\right)}}{\left[1+c_2^{-2}\left(1+c_1^2\right)-c_2^{-2}\lambda_h\right]}-c_2^{-2}\lambda_h
\end{aligned}$$

We go at the order $\sqrt{\lambda_h}$

$$c \quad = \quad \frac{\sqrt{\lambda_h}}{\sqrt{\left(1+c_1^2\right)\left(1+c_2^{-2}\left(1+c_1^2\right)\right)}} \qquad \text{and} \qquad 1+d \quad = \quad \frac{c_2^{-2}\left(1+c_1^2\right)\sqrt{\lambda_h}}{\sqrt{\left(1+c_1^2\right)}\sqrt{1+c_2^{-2}\left(1+c_1^2\right)}}$$

The variance 36 $\quad \Omega \quad = \quad [c+d]^{-1}\begin{pmatrix} d\left(1+c_1^2\right) & -(1+d)c_2^2 \\ -c\left(1+c_1^2\right) & (-1+c)\,c_2^2 \end{pmatrix}$

The PT measure

$$PT_1=\frac{1}{1+c_2^{-2}\left(1+c_1^2\right)} \quad \text{and} \quad PT_2=\frac{c_2^{-2}\left(1+c_1^2\right)}{1+c_2^{-2}\left(1+c_1^2\right)}$$

The information share bounds The total IS is $\psi\Omega\psi' = -[c+d]^{-1}c^2\left(1+c_1^2\right)\left(1+c_2^{-2}\left(1+c_1^2\right)\right)$

The bounds for market 1 are

- $IS1u=\frac{KD^2}{\psi\Omega\psi}\frac{\left(1+c_1^2\right)}{d}$ and $IS1l=-\frac{KD^2}{\psi\Omega\psi}\left(1+c_1^2\right)\left(\frac{c+d}{-1+c}\right)$

The bounds for market 2

- $IS2u = \frac{K}{\psi\Omega\psi}\frac{c^2}{(-1+c)}\left(1+c_1^2\right)^2 c_2^{-2}$ and $IS2l = -\frac{K}{\psi\Omega\psi}\frac{(c+d)}{d}\left(1+d\right)^2 c_2^2$

To summarize we have $K = [c+d]^{-1} = \left[D\left(1+c_2^{-2}\left(1+c_1^2\right)\right)-1\right]^{-1}$, $c = D$,
$1+d = c_2^{-2}\left(1+c_1^2\right)D$, $\psi\Omega\psi = -K\lambda_h$
with $D = \sqrt{\lambda_h}\left(\sqrt{\left(1+c_1^2\right)\left(1+c_2^{-2}\left(1+c_1^2\right)\right)}\right)^{-1}$ and $D \overset{h\to 0}{\longrightarrow} 0$

$$
\begin{array}{rclcccc}
IS1u & = & -\frac{1}{c_2^{-2}\left(1+c_1^2\right)D-1}\times\frac{1}{\left(1+c_2^{-2}\left(1+c_1^2\right)\right)} & = & \frac{1}{1+c_2^{-2}\left(1+c_1^2\right)} & = & PT_1 \\
IS1l & = & \left(\frac{D\left(1+c_2^{-2}\left(1+c_1^2\right)\right)-1}{D-1}\right)\frac{1}{1+c_2^{-2}\left(1+c_1^2\right)} & = & \frac{1}{1+c_2^{-2}\left(1+c_1^2\right)} & = & PT_1 \\
IS2u & = & \frac{-\left(1+c_1^2\right)c_2^{-2}}{(-1+D)}\times\frac{1}{\left(1+c_2^{-2}\left(1+c_1^2\right)\right)} & = & \frac{\left(1+c_1^2\right)c_2^{-2}}{1+c_2^{-2}\left(1+c_1^2\right)} & = & PT_2 \\
IS2l & = & \frac{1}{\left(1+c_2^{-2}\left(1+c_1^2\right)\right)}\frac{\left[D\left(1+c_2^{-2}\left(1+c_1^2\right)\right)-1\right]}{Dc_2^{-2}\left(1+c_1^2\right)-1}c_2^{-2}\left(1+c_1^2\right) & = & \frac{\left(1+c_1^2\right)c_2^{-2}}{1+c_2^{-2}\left(1+c_1^2\right)} & = & PT_2
\end{array}
$$

We obtain also here that the IS bound and PT are similar at high frequency.

C.5 Proof of Proposition 3

Proof. If $1 \geq \alpha_1$ then

$$
\begin{aligned}
\kappa &= -\frac{\sigma^2 h}{2} + \frac{\sigma h}{2}\sqrt{4c_1'^2 c_2'^2 h^{\alpha_1}\left(c_1'^2 h^{\alpha_1-\alpha_2}+c_2'^2\right)^{-1}} \simeq -\frac{\sigma^2 h}{2} + \sigma c_1' c_2' h^{0.5+\alpha_1}\left(c_1'^2 h^{\alpha_1-\alpha_2}+c_2'^2\right)^{-1/2} \\
&\longrightarrow 0 \\
K &= \left[1-\kappa h^{-\alpha_1}\left(c_1'^{-2}+c_2'^{-2}h^{\alpha_1-\alpha_2}\right)\right]^{-1} \\
&= \left[1-\left(-\frac{1}{2}\sigma^2 h+\sigma c_1' c_2' h^{0.5+\alpha_1}\left(c_1'^2 h^{\alpha_1-\alpha_2}+c_2'^2\right)^{-1/2}\right)h^{-\alpha_1}\left(c_1'^{-2}+c_2'^{-2}h^{\alpha_1-\alpha_2}\right)\right]^{-1} \\
&\backsimeq \left[1-\left(-\frac{1}{2}\sigma^2 h h^{-\alpha_1}+\sigma c_1' c_2' h^{0.5+\alpha_1}h^{-\alpha_1}\left(c_1'^2 h^{\alpha_1-\alpha_2}+c_2'^2\right)^{-1/2}\right)\left(c_1'^{-2}+c_2'^{-2}h^{\alpha_1-\alpha_2}\right)\right]^{-1} \\
&\backsimeq \left[1+\frac{1}{2}\sigma^2 c_1'^{-2}h^{1-\alpha_1}-\sigma c_1 c_2 h^{0.5}c_2'^{-1}c_1'^{-2}\right] \\
&\longrightarrow 1
\end{aligned}
$$

$\square$