Fundamental Bubbles in Equity Markets*

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Abstract

We use a dynamic affine term structure framework to price equity and bonds jointly, and investigate how prices are related to a set of macro factors extracted from a large dataset of economic time series. We analyze the discrepancies between market and model implied equity prices and use them as a measure for bubbles. A bubble is diagnosed over a given period whenever the discrepancies are not stationary and impact the underlying economy consistently with the literature's findings, increasing over the shorter term economic activity before leading to a net loss in it. We perform the analysis over 3 major US and 3 major European equity indices over the 1990-2017 period and find bubbles only for two of the US equity indices, the S&P500 and the Dow Jones.

JEL Codes: G12, C58, E44.

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1 Introduction

Equity bubbles are often rumored but rarely accurately measured. In essence, a bubble is a large and long lasting departure of the price of an asset from its "fundamental valuation". This last term can have a large number of meanings, which are dependent on the econometric model used. Here, we propose to use the empirical connection existing between earnings growth rates and macro-economic fundamentals to measure the fundamental valuation of equity indices. A discrepancy between actual and fundamental valuation that is found to be non-stationary and that induces the well-documented consequences of a bubble in the underlying economy is then diagnosed as a bubble. We find that a bubble has driven the behavior of both the S&P500 and the Dow Jones, but not the Russell 2000 and the MSCI Europe and Germany indices over the 1990-2017 period.

Following Gürkaynak (2008), there exists a long lasting literature that aims at measuring, predicting and evaluating the consequences of bubbles in financial markets. When it comes to equities markets, a significant portion of it is based on an expected dividends model, in a similar fashion to Shiller's various bound test (Shiller (1981)): when equity prices are significantly above their fundamental valuation, a bubble is diagnosed. When such a measure is based on a dividend expectation model, its is usually referred to as "rational bubble" (see Gürkaynak (2008)). Four types of bubble detection model can be found in the literature: first, the various bound test of Shiller (1981) and LeRoy and Porter (1981). The intuition is that if actual equity prices are more volatile than what is implied by realized dividends, it indicates that a bubble is driving the prices. This research direction has been abandoned for various reasons detailed in Gürkaynak (2008), among of which are implementation difficulties. A second approach is detailed in West (1988): comparing a model implied connection between dividends and prices and its empirical counterpart, a test is built in order to decide whether prices display a bubbly behavior. Here, the model specification is essential and time varying interest rates is one potential source of the problem to this approach. A third stream of articles exploit the fact that bubbles are likely to create non-stationarity in the time series of asset prices. Grossman and Diba (1988) build their test on the fact that dividends and prices should be cointegrated in the absence of a bubble. The periodicity of the rise and burst of bubbles can however lead to fail to find a bubble when there is one, as the global behavior of the stock price-dividend relationship remains stationary over the long run, as explained in Evans (1991). Other unit-root tests have also been proposed in the literature, such as, for instance, the Markov-Switching based test proposed in Hall et al. (1999). Finally, a last strand of literature aims at finding bubbles without relying on a economically motivated fundamental model, arguing that locally a bubble would create price movement that are odd given the usual functional form of the relationship between time and prices. One of the examples of this approach includes Sornette et al. (2009).

As Gürkaynak (2008) concludes his review of literature: "For every test that "finds" a bubble, there is another paper that disputes it. The finding of a bubble, at best, suggests that the data is either consistent with a bubble or a myriad of other extensions of the standard model". One of the aims of this work is to shed new light on this debate and its findings, and to exploit an affine pricing model and macroeconomic fundamentals: we intend on measuring bubbles that reflect a departure of equity prices from what would be the pure reflection of a macroeconomic data set. From this perspective, our approach is different from the cointegration/unit-root/citewest1988bubbles tests: it builds on a empirically grounded modeling of the connection between fundamentals to earning growth and macro data and then exploits this connection to compute a fundamental price to equities.

We employ an Affine Term Structure (ATS hereafter) model setup, first developed by Duffie and Kan (1996) and characterized in Duffie et al. (2003) and Dai and Singleton (2000). The finance literature on modeling the term structure of interest rates and on the pricing of equities have evolved almost independently. The bond literature is generally aimed at modeling rates, swaps, caps and floors and other derivatives using the term structure structure of rates. The equity stream of literature tries to unearth cross-sectional factors or time series stylized facts related to equity prices. It has been however well empirically documented that equity and bonds markets are interrelated: the pricing of equity indices should impact the pricing of bonds and vice versa.

Tractability of affine term structure models led to their implementation in a variety of asset classes and applications to multitude of topics. The first major applications of dynamic affine term structure (DATS) models focused on credit risk and the term structure of interest rates. Interest rates have been analyzed under macroeconomic lenses most notably in Ang and Piazzesi (2003), Cochrane and Piazzesi (2009), Rudebusch and Wu (2008), Wu (2001), Kim and Wright (2005) and more recently in Joslin et al. (2014), Bauer and Diez de los Rios (2012) and Bauer and Rudebusch (2016) . Key accepted facts about the yield term structure include strong predictability of excess bonds' returns and a strong link between the bond risk premia and the level of the term structure, and the presence of unspanned risks.

Another strand of the ATS literature focuses on the joint dynamics between the interest rates term structure and the price of equities. Among many papers, Lemke and Werner (2009) , Ang and Ulrich (2012), Lettau and Wachter (2011), Bekaert and Grenadier (1999), and Bansal and Yaron (2004) analyze rates and equities simultaneously, shedding light on a number of macroeconomic issues. Despite the fact that these articles focus on various topics, they consistently find a strong connection between expected equity returns and the variation in output and inflation. Also, risk premia time series and economic business cycles are found to be strongly interrelated when focusing on indices, as opposed to individual stocks. Dividend yields have also been found to have a predictive powewr over equity returns. A majority of these papers also include macroeconomic variables into their state vector, most notably measures of output and inflation. A more recent strand of the equities ATS literature (Belo et al. (2015), Van Binsbergen and Koijen (2017), Van Binsbergen et al. (2012)) deals with producing empirically observed downward sloping term structure of dividend strips. Also, a majority of researchers employ Gordon-type formulas to obtain model solutions for the price to dividend ratio and its associated transforms, which is the infinite sum of the price of dividends strips[1] .

This article aims at relying on their findings to test for bubbles in equity prices using the departure of equity market prices from the an ATS model-implied index price. Here, we want to highlight potential departures of equity indices' prices from the coincident fundamental picture, as proxied by a set of macroeconomic factors. Throughout the article, bubbles are defined as a long lasting departure between a fundamental valuation of equity indices' prices and their actual market price. This fundamental valuation is obtained by combining a set of macro-economic factors and an affine equity pricing model. The model exploits the existing empirical connection between economic factors and the growth in equities' earnings: the price of equity indices is derived from it. Then, by testing for the stationarity of the discrepancy between prices of indices and their fundamental valuation counterparts, we diagnose bubbles both for the S&P500 and the Dow Jones over the 1990-2017 period. Finally, we test whether this discrepancy generates the expected consequences of a bubble over the underlying economy, first boosting economic activity before later weighing on it severely. Both the S&P500 and Dow Jones results exhibit such results.

The rest of the article is organized as follows: Section 2 provides an analysis of the connection between a set of macroeconomic factors and earnings growth, and the conclusion from this section will serve as a backbone for the building of our macro-finance model. Section 3 introduces the dynamic affine term structure model, discussing various aspects of it, and deriving solutions to equity and bond prices. Section 4 details our empirical findings. Finally, Section 5 summarizes the findings and concludes.

2 Empirical motivation

This section aims at providing the empirical elements that motivate the modeling approach that will be presented in the forthcoming portion of this work. The objective of this article is to model and estimate a valuation model of stock markets that would be purely based on macroeconomic data, and

[1] Dividend strip refers to a price of a single dividend claim k periods into the future.

treating the departure from it as bubble indicator. This implies two steps: first, we need to define the macro data that we intend on using as the fundamental information set; then, we need to hypothesize a connection between this information set and the price of stocks.

2.1 Macroeconomic factors building

When it comes to the first of these two elements, there are various ways of achieving this task. One is to decide to focus on specific economic phenomenons such as Gross Domestic Product (GDP) growth or changes in the price level. This has been the approach used in Ang and Piazzesi (2003) and Ang et al. (2006) to analyze the macroeconomic fundamentals to the term structure of interest rates. Other attempts using inflation and consumption growth are Eraker (2008) and Swanson et al. (2014) focusing on the pricing of equities. Another path is to consider that (1) there are many potential economic factors that could be entering the pricing kernel to estimate the asset prices and (2) that GDP growth and Consumer Price Indices (CPI) are potentially partial and noisy variables given the importance of their revisions. On the latter point, Faust et al. (2005) quantifies the uncertainty triggered by these measurement issues. Using "data-rich" environment paves a natural way out of both these issues: by relying on a large set of economic data, common underlying factors can be estimated and then identified through a correlation analysis. Articles using this type of approach in order to connect such factors to asset returns and prices include Mönch (2008a) and Ghysels et al. (2014) in the case of bonds, while Beber et al. (2015) uses similar measures to explain stock market volatility. Flannery and Protopapadakis (2002), for instance, find that equity returns are not influenced by the publication of GDP growth in the US once a large enough dataset of economic time series is considered. From this perspective, connecting key economic fundamentals to broader sets of economic indicators looks particularly promising and we will follow this approach.

The approach of extracting a few significant common factors from a large set of macroeconomic time series was advocated by, among others, Bernanke et al. (2005). They argue that macroeconomic aggregates, such as inflation and output, might not be observable to a researcher or policy maker. Instead, they observe a set of macroeconomic time series that are noisy realizations of economic concepts related to output and inflation. Therefore, Bernanke et al. (2005) propose to extract several common factors and then study the mutual dynamics of key economic aggregates and dynamics of monetary policy, in particular using a joint VAR of macro factors and monetary policy instrument (they label this approach 'Factor-Augmented VAR' or 'FAVAR').

We rely on a set of 186 economic time series that cover different types of economic phenomenons, including growth, inflation, employment and commodities. All data series are listed in Tables 9, 10 and 11. The dataset also covers different types of countries. The constraint that has been used in the data selection process has been to find economic data-series that started at least in 1990: the original list of time series was larger and has been reduced to satisfy this constraint. In addition, we use the initially published value for each data when available in order to best reflect the available information to market participant. Each data is then z-scored by subtracting its long-run mean and dividing this difference by the standard deviation of each time series. Finally, a Principal Component Analysis (PCA) is then ran on this dataset of z-scored time series and factors are computed using the eigenvectors obtained from the covariance among all time series. The eigenvalues for the first 20 factors are displayed on Figure 2. As expected, a large part of the information in this dataset can be concentrated into a limited number of factors. The first five factors incorporate 52% of the information spanned by the dataset and the first factor retains 24% of it. In order to decide on the proper number of factors to be retained to describe the economy, we rely on both the investigation of Figure 2 and on the tests proposed in Alessi et al. (2010) and Trapani (2017). The tests recommend three factors[2]. Table 8 presents the 20 largest correlations between each of the three factors and the

[2]One of the advantages of the above factors specification is that they are fully observable and have an approximate economic interpretation. Piazzesi (2010) reports that the model implied dynamics for macro variables (i.e. latent factor estimated from the model) are often disconnected from the dynamics of their historical observable counterparts. Also, latent factors require computationally burdensome estimation techniques that account for the joint distribution of yields or price to dividend ratios

underlying dataset. Factor 1 is strongly correlated to growth indicators. Factor 2 is largely connected to inflation metrics and sources. Factor 3 opposes European data series to US ones.

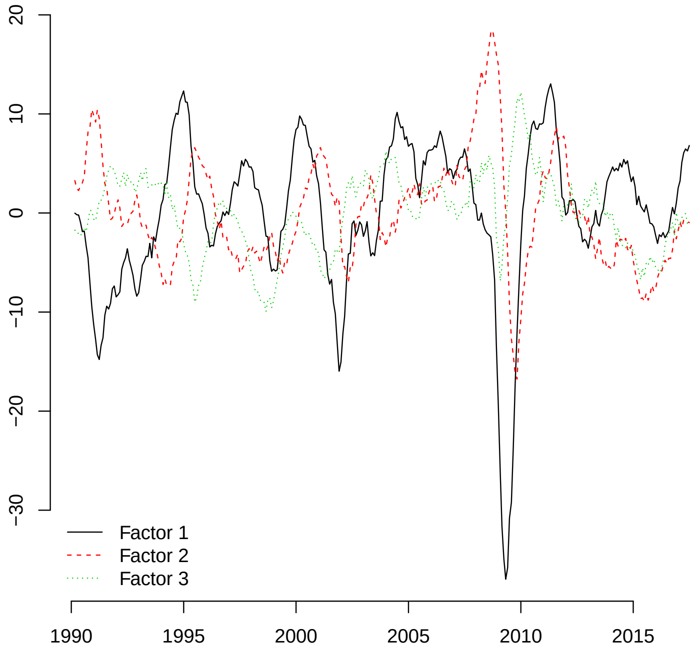

Figure 1: Time-series evolution of the Principal Components factors over the 1990-2017 period.

2.2 Initial data analysis on the connection between economic factors and earnings and dividend growth

As a second step, we present preliminary regressions between the three macro factors, earning growth, dividend growth and dividend payout ratio. The aim is to analyze which relationship between equity prices and the macroeconomic factors is the most natural and consistent across different markets. In order to broaden the perspective of these investigations, we rely on two markets: the US one with the S&P500, Russell and Dow Jones indices and the European one with the MSCI indices for Europe, Germany and France. For each of those indices, we run regressions of their earnings growth, dividend growth and dividend payout ratio (that is the ratio between dividends and earnings) on the macroeconomic factors derived previously. The data was obtained from Bloomberg and covers the 1990-2017 period and has a monthly frequency. As a macro factor, we also include the local short rate. Both in the case of the US and Europe, the short rate is approximated by the corresponding 3-month government yields (using German yields in the case of Europe).

The results of the regressions are presented in Table 1. The final objective of this table is to decide whether earnings or dividends have the strongest connection to macroeconomic fundamentals, the

and the state vector which can make our task significantly more difficult (see Ang and Piazzesi (2003) and Kim (2007) on those difficulties). Finally, specifications with latent factors are not always globally identifable and this issue in itself was closely scrutinized in the academia (Collin-Dufresne et al. (2008), Christensen et al. (2011), Joslin et al. (2011) and Hamilton and Wu (2012)).

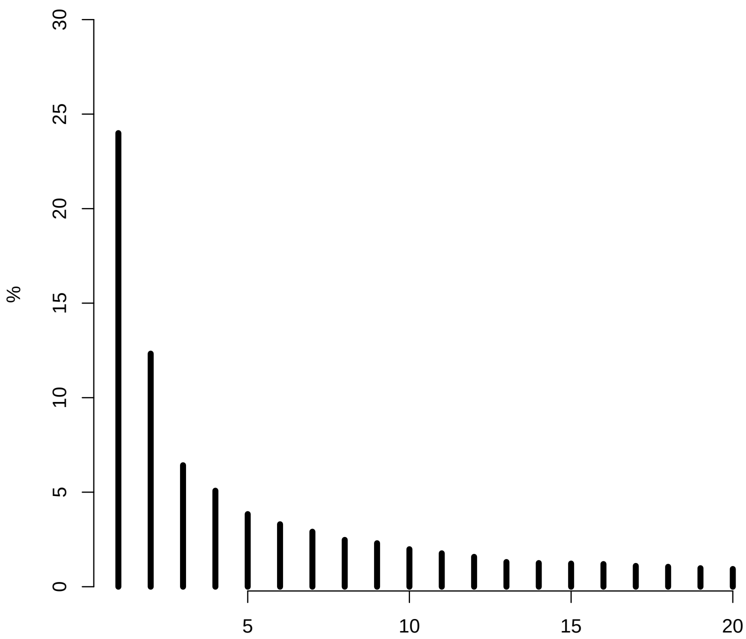

Figure 2: First rescaled first 20 eigenvalues associated to the Principal Component Analysis of the macro dataset.

link between the two being the dividend payout ratio. The investigation of the table shows consistent results across the six equity indices: on average, the R-squared coefficients obtained from the regression of earnings growth are higher than the one obtained with dividend growth, and the loadings of macroeconomic factors are more significant. Factor 1 is positively related to both when Factor 2 is positively related to dividend growth and negatively to earnings growth: inflation seems to be positive for dividends but not for earnings. The opposite is found for Factor 3 that is negatively related to dividend growth and positively to earnings growth. The dividend payout ratio is found to be strongly connected to earnings which is consistent with the idea of dividend smoothing: firms have a tendency to reduce their payout ratio in periods of growth and to increase it in periods of recession as illustrated among others in Leary and Michaely (2011) and Chen et al. (2012). Table 2 shows regression results when the payout ratio is the dependent variable and earnings growth the independent one. The average R-squared coefficient lies in the 80% region, and the relationship is negative and statistically significant, implying that higher earnings induce a lower distribution rate: another manifestation of dividend smoothing. Interestingly, the instantaneous relationship between earnings growth rates and the short rate is never found to be statistically significant.

These descriptive statistics seem to point in the direction of a model based on a connection between earnings growth rate and the macroeconomic factors, rather than dividend growth rates, as is it commonly done in ATS equity literature. The dividend growth rate will then be obtained by using the negative relationship between the dividend payout ratio and earning growth. Namely, the (log) earnings growth rates g_t are assumed to be affine in the state vector X_t

$$g_t = \gamma_0 + \gamma_1 X_t. \tag{1}$$

Exploiting the strong negative relationship between the payout ratio's growth rate and the earnings' growth rates

$$c_t = \alpha_0 + \alpha_1 g_t \tag{2}$$

leads to the following specification of dividend growth rates

$$d_t = \ln\frac{D_t}{D_{t-1}} = \ln\frac{E_t C_t}{E_{t-1} C_{t-1}} = c_t + g_t = \alpha_0 + (1+\alpha_1)\,\gamma_0 + (1+\alpha_1)\,\gamma_1 X_t =: \omega_0 + \omega_1 X_t \tag{3}$$

where $\omega_0 = \alpha_0 + (1+\alpha_1)\,\gamma_0$ and $\omega_1 = (1+\alpha_1)\,\gamma_1$ are obtained from the earnings' and payout ratio's growth's loadings on the state vector factors. Importantly, d_t is affine in the state vector X_t, as in equation (31), thus the results of the ATS framework for equity pricing still apply.

In what follows, the local short rate will not be used as an explanatory variable in the earnings growth dynamics due to its insignificance. The next section presents the estimates of DATS model derived from these preliminary analyses.

Table 1: Regression analysis between the macroeconomic factors and dividend growth, earning growth and dividend payout ratio in the case of the US and Europe.

		S&P500	Dow Jones	Russell	MSCI Europe	MSCI Germany	MSCI France
Dividend growth	Intercept	0.022*	0.016*	0.018	0.007	0.011	0.005
	Local short rate	-0.003*	0	-0.001	0.003	-0.001	0.003
	Factor 1	0.001*	0.001*	0.003*	0.003*	0.003	0.002
	Factor 2	0.001*	0.001	0.001	0	0.002	0.001
	Factor 3	-0.002*	-0.001	-0.001	-0.002	-0.003	-0.002
	R2	0.32	0.24	0.17	0.25	0.06	0.08
Earnings growth	Intercept	0.009	0.002	0.014	-0.019	-0.009	-0.002
	Local short rate	0.002	0.005	-0.002	0.012	0.02	0.009
	Factor 1	0.004*	0.002	0.008*	0.013*	0.014*	0.011*
	Factor 2	-0.003*	-0.004*	-0.004	-0.01*	-0.01*	-0.008
	Factor 3	0.003*	0.006*	0.007	0.014*	0.016*	0.009
	R2	0.432	0.150	0.139	0.314	0.299	0.162
Dividend payout ratioo	Intercept	0.345*	0.398*	0.435*	0.722*	0.384*	0.64*
	Local short rate	0.018*	-0.017*	-0.013*	-0.007	0.499*	0.11*
	Factor 1	-0.008*	-0.005*	-0.009*	-0.031*	-0.059*	-0.02*
	Factor 2	-0.004*	0.004*	-0.002	-0.02*	-0.088*	-0.024
	Factor 3	0.009*	0.003	0.011*	0.021*	-0.044	-0.012
	R2	0.402	0.539	0.559	0.479	0.493	0.143

This table displays the outcome of the regression between dividend growth, earnings growth and dividend payout ratio on the macroeconomic factors and the local short rate.

Table 2: Regression outcome of the dividend payout ratio on the growth in earnings.

		S&P500	Dow Jones	Russell	MSCI Europe	MSCI Germany	MSCI France
Dividend payout ratio	Intercept	0.012*	0.018*	0.016*	0.013*	0.014	0.012
	Earnings growth rate	-0.98*	-1.021*	-0.935*	-0.958*	-1.057*	-0.975*
	R2	0.855	0.942	0.897	0.955	0.793	0.931

The table reports OLS regression estimates of $c_t = \alpha_0 + \alpha_1 g_t$, where c_t and g_t denote payout ratio and earnings logarithmic growth rates, respectively. Starred estimates denote parameters that are significant at 10% level.

3 The model

In this section we detail the asset pricing model linking macroeconomic factors to the price of stocks indices and bonds. We first detail the short rate dynamics, the pricing kernel and then deduce from them the term structure of bonds' yields. Finally, we derive the price of stock indices under this economy.

3.1 Short rate dynamics

A usual approach to describing monetary policy in an asset pricing model is to make the short term rate r_t an affine function of X_t, the vector containing the state variables describing our economy at time t:

$$r_t = \delta_0 + \delta_1' X_t. \tag{4}$$

Usually, with such a modeling approach, X_t only contains the factors driving the economy F_t. In an affine term structure model, using equation (4) implies that Central Banks have no impact over the economy, which is a questionable assumption. This also contradicts the empirical evidence that term structure movements can predict macroeconomic activity as pointed out in Ang and Piazzesi (2003) and consistently. Harvey (1988) and Estrella and Hardouvelis (1991) highlight this connection between yields and the economy. This drawback is addressed in Mönch (2008b), Hördahl et al. (2006) and Jardet et al. (2013) by including the short rate into the set of factors driving the economy. The joint dynamics of these factors and the monetary policy instrument is then modeled in a Vector Autoregressive (VAR) setting. As a result, yields are driven by both movements in the short rate and the macroeconomic factors, and there is a feedback mechanism between monetary policy and the economy. Our short rate specification follows this approach:

$$X_t = \begin{pmatrix} F_t \\ r_t \end{pmatrix} = \mu + \Phi \begin{pmatrix} F_{t-1} \\ r_{t-1} \end{pmatrix} + \Sigma \varepsilon_t \tag{5}$$

where μ is a $(k+1)$ vector of constants, Φ is a $(k+1) \times (k+1)$ matrix of loadings, $\varepsilon_t \sim IID\ N(0, I_{k+1})$ such that $\Sigma'\Sigma = \Omega$, where Ω is the variance covariance matrix of economy shocks and k is the number of fundamental macroeconomic factors used ($k+1$ also accounts for the short rate in the state vector vector). This can easily be generalized to a VAR(p) structure.

As it is common in the ATS literature, the short rate is linked to the state vector through the last row in equation (5) which can be rewritten as

$$r_t = \mu_r + \Phi_F F_{t-1} + \Phi_r r_{t-1} + \Sigma_r \epsilon_t \tag{6}$$

where μ_r is the last entry in μ, Φ_F is the row of Φ excluding the last entry, Φ_r is the last entry of the last row of Φ, and Σ_r is the last row of matrix Σ. We further denote $\delta_r' = \left(0_{k\times 1}', 1\right)'$ a unit vector that picks short rate process from the state vector, i.e. $r_t = \delta_r' X_t$.

3.2 The pricing kernel

In order to price equities and bonds, we use the assumption of no-arbitrage[3] to guarantee the existence of the risk-neutral probability measure Q such that the discount price of any asset V_t that does not produce cash flows from time t to $t+1$ satisfies:

$$V_t = E_t^Q(e^{-r_t} V_{t+1}), \tag{7}$$

where the expectation is under the risk-neutral probability measure Q. The Radon-Nikodym density which links the risk-neutral measure with data-generating measure through Girsanov's theorem is denoted by ξ_t. Under this probability measure, any stochastic variable Z_t satisfies:

$$E_t^Q(Z_{t+1}) = E_t(\xi_{t+1} Z_{t+1})/\xi_t \tag{8}$$

The assumption of no-arbitrage and the assumption of existence of ξ_t (and hence Q) are equivalent and allow to price any cash-generating asset in the economy, including equity and bonds.

In line with ATS literature, we assume that the density ξ_{t+1} follows the following log-normal process:

$$\xi_{t+1} = \xi_t \exp\left(-\frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\right) \tag{9}$$

where λ_t is a vector of time-varying market price of risk associated with the sources of uncertainty ε_t. We assume that Novikov's condition is satisfied[4]. In line with the literature (Constantinides (1992),

[3] Technical condition are described in Harrison and Kreps (1979).

[4] For technical details, see Appendix D and E in Duffie (2004). By construction, ξ_t is a strictly positive martingale under Novikov's condition with $\xi_0 = 1$, implying that P and Q are equivalent and Q is a well defined probability measure.

Fisher and Gilles (1998), Ang and Liu (2001), Duffee (2002), Dai and Singleton (2002) and others), we assume that the market price of risk λ_t is an affine process:

$$\lambda_t = \lambda_0 + \lambda_1 X_t \tag{10}$$

where λ_0 is a k-dimensional vector and λ_1 is a $k \times k$ matrix of parameters[5]. This corresponds to the essentially affine class of models where X_t is affine under both physical and risk-neutral measures whereas $\lambda_t' \lambda_t$ is not affine in X_t, which however does not affect the pricing of bonds or equities[6]. Equations (9) and (10) allow for shocks in the risk factors, both macro and latent, to impact the market price of risk ξ_{t+1} and therefore determine how factor shocks affect yields and the price of equities. Importantly, equation 10 also implies that the market price of risk itself is completely spanned by the chosen factors.

The nominal pricing kernel, or stochastic discount factor (SDF), M_t is defined as

$$M_{t+1} = \exp(-r_t) \frac{\xi_{t+1}}{\xi_t}. \tag{11}$$

Substituting equations (4) and (9) into equation (11), we obtain:

$$M_{t+1} = \exp\left(-\frac{1}{2}\lambda_t' \lambda_t - \delta_0 - \delta_1' X_t - \lambda_t' \varepsilon_{t+1}\right). \tag{12}$$

Denoting the log real stochastic discount factor $m_t := \ln M_t$, we obtain:

$$m_{t+1} = -\frac{1}{2}\lambda_t' \lambda_t - \delta_0 - \delta_1' X_t - \lambda_t' \varepsilon_{t+1}, \tag{13}$$

where $\delta_0 = 0$ and $\delta_1 = \delta_r$ under the short rate specification proposed in equation (6).

Under the condition of no-arbitrage, the price at time t of an asset with a nominal payoff P_{t+1} at time $t+1$ is given by:

$$P_t = E_t^Q(\exp(-r_t) P_{t+1}) = E_t(M_{t+1} P_{t+1}). \tag{14}$$

Alternatively, the total gross return R_{t+1} of any nominal asset must satisfy:

$$E_t(M_{t+1} R_{t+1}) = 1. \tag{15}$$

3.3 Nominal zero-coupon bonds

Having state space dynamics and nominal pricing kernel in place, we can price any nominal assets in the economy. Equation (14) allows to price nominal assets given information on their cash flows and holds both for equities and bonds. Therefore, the price of the bond maturing in τ time periods at time t satisfies

$$P_t(\tau) = E_t\left(M_{t+1} P_{t+1}(\tau - 1)\right) = E_t M_{t+\tau}. \tag{16}$$

With the specification described here[7], it follows that zero-coupon bond prices are exponential affine functions of the state vector[8]:

$$P_t^n = \exp\left(A_n + B_n' X_t\right), \tag{17}$$

[5]Intuitively, with the state space dynamics in equation (5), λ_t must be an affine function of X_t as in equation (10) for the drift term of X_t to remain affine in X_t under the risk-neutral probability measure as well. The diffusion of the state vector is the same under both measures .

[6]Essentially affine models span the completely affine model described in Dai and Singleton (2000). The specification of the market price of risk is studied in detail in Cheridito et al. (2007).

[7]r_t is an affine function of X_t, X_t is also affine under the risk-neutral probability measure due to the specification of the X_t process as a Gaussian VAR, the SDF and the MPR specifications.

[8]For details, see, e.g., Duffie et al. (2003),Ang and Piazzesi (2003) among others.

where coefficients A_n and B_n are deterministic functions of time and satisfy the following discrete system of ODEs, with the n subscript representing the number of periods until maturit):

$$A_n = A_{n-1} + B'_{n-1}(\mu - \Sigma\lambda_0) + \frac{1}{2}B'_{n-1}\Sigma\Sigma' B_{n-1} - \delta_0 \tag{18}$$

$$B'_n = B'_{n-1}(\Phi - \Sigma\lambda_1) - \delta'_1, \tag{19}$$

which under the short rate specification presented in equation (6), imposing $\delta_0 = 0$ and $\delta_1 = \delta_r$, becomes

$$A_n = A_{n-1} + B'_{n-1}(\mu - \Sigma\lambda_0) + \frac{1}{2}B'_{n-1}\Sigma\Sigma' B_{n-1} \tag{20}$$

$$B'_n = B'_{n-1}(\Phi - \Sigma\lambda_1) - \delta'_r. \tag{21}$$

The associated boundary conditions are

$$A_1 = 0 \tag{22}$$

$$B_1 = -\delta^r \tag{23}$$

and stem from the fact that the price of a maturing bond must be equal to 1. The derivations of different equations and boundary conditions are presented in Appendix A.

We, furthermore, note that

$$\mu^Q = \mu - \Sigma\lambda_0 \tag{24}$$

$$\Phi^Q = \Phi - \Sigma\lambda_1 \tag{25}$$

are the new parameters of the corresponding risk-adjusted dynamics of the state vector X_t and they appear in equations (18) and (21).

The continuously compounded yield of of an n-period zero-coupon bond is given by

$$y_t^n = -\frac{\ln P_t^n}{n} = \bar{A}_n + \bar{B}'_n X_t \tag{26}$$

where $\bar{A}_n = -\frac{A_n}{n}$ and $\bar{B}_n = -\frac{B_n}{n}$.

3.4 Equities

Whereas zero-coupon bonds guarantee payment of 1 unit (whether real or nominal) in the future, holders of equity assets benefit from price appreciation and dividend payments. Under risk-neutral probability measure Q, the nominal equity price V_t must satisfy

$$V_t = E_t^Q \left[e^{-r_t} \left(V_{t+1} + D_{t+1} \right) \right], \tag{27}$$

which recursively generalizes to

$$V_t = E_t^Q \left[\sum_{n=1}^{\infty} e^{-\sum_{k=1}^{n} r_{t+k-1}} D_{t+n} \right]. \tag{28}$$

The expectations in equations (27) and (28) are under the risk-neutral probability measure Q associated with the nominal pricing kernel introduced previously. In this setting, equity prices represent a claim on an infinite stream of dividend strips. Normalizing equity prices by dividends and using the growth rates of real dividends

$$d_{t+1} := \ln \frac{D_{t+1}}{D_t}, \tag{29}$$

we can rewrite equation (28) as follows[9]:

$$\frac{V_t}{D_t} = E_t^Q \left[\sum_{n=1}^{\infty} \exp\left(\sum_{k=1}^{n} d_{t+k} - r_{t+k-1} \right) \right]. \tag{30}$$

Using the term structure of nominal interest rates, the stream of dividends is then discounted in order to obtain equitiy prices.

Consistently with the literature, we assume that the logarithmic dividend growth rate is an affine function of the state vector and is given by

$$d_t = \omega_0 + \omega_1 X_t. \tag{31}$$

Similar to the pricing of bonds, in Appendix B we show that under certain restrictions, the price of a single normalized dividend strip V_t^d that pays D_{t+n} at time $t+n$ is given by

$$V_{t,n}^d = E_t^Q \left[e^{-\sum_{k=1}^{n} r_{t+k-1}} \frac{D_{t+n}}{D_t} \right] = E_t^Q \left[\exp\left(\sum_{k=1}^{n} d_{t+k} - r_{t+k-1} \right) \right] = \exp\left(a_n + b_n' X_t\right), \tag{32}$$

which implies that both bonds and dividend strips are exponential affine functions of the state vector X_t. The deterministic maturity-varying parameters a_n and b_n satisfy the following system of ordinary difference equations:

$$a_{n+1} = a_n + \omega_0 + (\omega_1 + b_n)' \mu^Q + \frac{1}{2} (\omega_1 + b_n)' \Sigma\Sigma' (\omega_1 + b_n) \tag{33}$$

$$b_{n+1} = \Phi^{Q'} (\omega_1 + b_n) - \delta^r \tag{34}$$

where δ^r is the unit vector that picks the short rate process from the state vector (i.e. $r_t = \delta^r X_t$), while μ^Q and Φ^Q are given by equations (24) and (25).

The boundary conditions are given by[10]:

$$a_1 = \omega_0 + \omega_1 \mu^Q + \frac{1}{2} \omega_1' \Sigma\Sigma' \omega_1 \tag{35}$$

$$b_1 = \Phi^{Q'} \omega_1 - \delta^r. \tag{36}$$

The fact that the price of the dividend strips are normalized by the current dividend implies that $V_{t,0}^d = 1$, which in turn implies $a_0 = 0$ and $b_0 = 0$. The price-dividend ratio thus admits the following form:

$$\frac{V_t}{D_t} = \sum_{n=1}^{\infty} \exp\left(a_n + b_n' X_t\right). \tag{37}$$

The difference between the systems of differential equations (33) and (34) vs. (18) and (21) is that the former now explicitly depends on the dividend growth rates parameters ω_0 and ω_1. If both of them are set to zero, the price of equities could be interpreted as a perpetual coupon bond.

We now propose to use this framework jointly with the macroeconomic factors presented in Section 2 in order to compute "fundamental price" to equities.

4 Empirical results

This section presents empirical results obtained with the model presented in the previous section. It starts by presenting the dataset of equity indices and interest rates used for the estimation and calibration. Then, the parameter estimation results are reviewed. Finally, we analyse the outcomes of the bubble estimation methodology across US and European markets.

[9] d_t is indexed one period ahead of the corresponding r_{t-1} due to the fact that r_t is a locally deterministic discount factor and is known at time t

[10] See Appendix B.

4.1 Dataset construction

We rely on two different datasets.[11] First, the dataset of economic indicators presented in the Section 2 gathers monthly data over the 1990-2017 period. From this dataset, consistently with has already been presented, we extract three macroeconomic factors that can be understood as a growth, an inflation and a Europe vs. US macro factor.

The second source of data we employ is a dataset of market data. It consists of US and European indices and rates. The US dataset contains the price of the following equity indices: the S&P500, the Dow Jones and the Russell 2000. Alongside the prices, the dataset also contains the estimates for dividends and earnings[12]. Given the nature of this data, we rely on a quarterly frequency dataset. In terms of rates, we retain the 3-month bill rate as a proxy to the short rate. In addition, our dataset contains the 1-, 2-, 5- and 10-year rates. The period covered starts in June 1990 and ends in June 2017. A second part of this dataset consists in similar data for the European market. We analyze the following European equity indices: the MSCI Germany, the MSCI France and the MSCI Europe indices. When analyzing each of the European equity indices, we use German government bonds rates with the same maturities as in case of the US. The European dataset is shorter than the US one, starting in March 1995 and ending in June 2017. The time series evolution of equity indices as well as of their price to dividend ratio is presented in Figure 3: unsurprisingly, the equity to dividend ratio creates the suspicion of a bubble around 2001, however, this is not true for all of equity indices. The S&P500 and the Dow Jones indices exhibit more pronounced signs of bubbles.

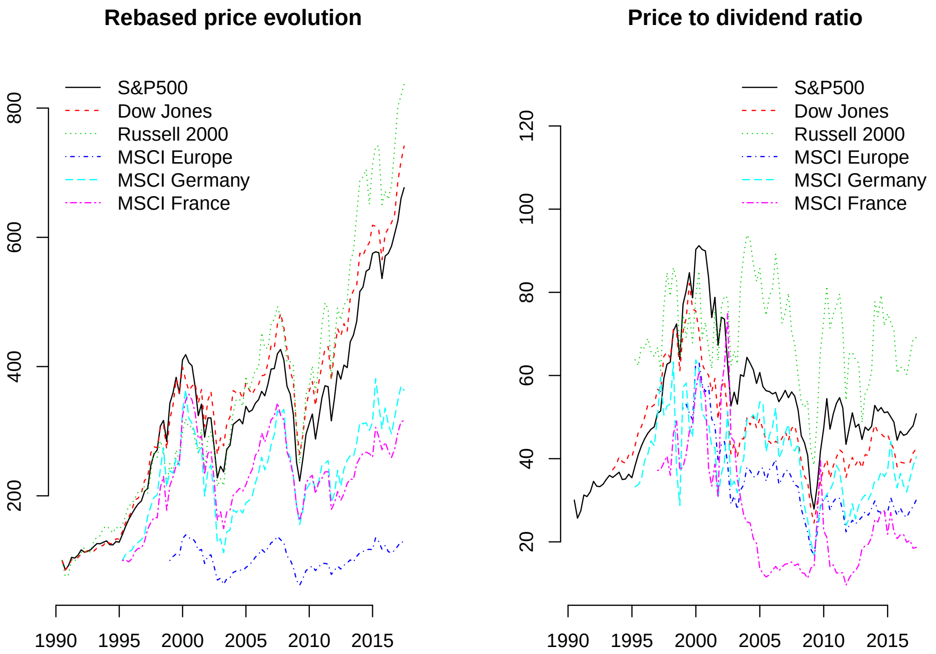

Figure 3: Time-series evolution of equity prices (left) and equity to dividend ratio (right) over the 1990-2017 period.

[11] All data used here have been obtained from Bloomberg© as a datasource.
[12] Here, we use BEST estimates.

4.2 Model estimation

Having the short rate and three fundamental factors extracted from a large dataset of macroeconomic variables, we estimate the term structure model of equities and rates simultaneously. We perform the estimation of the model using a two-step approach, following Ang and Piazzesi (2003), Mönch (2008b), Lemke and Werner (2009), Jardet et al. (2013), Monfort and Pegoraro (2007) and others.

In the first step, we run the OLS regression of VAR(p) model to obtain the estimates of (μ, Φ, Σ) that govern its dynamics under the data-generting P measure. We choose a VAR(1) parametrization based the usual information criterion and a certain need for a parsimonious model. Furthermore, we estimate the variance-covariance matrix Ω of the data generating process ε_t using the residuals obtained after performing the estimation. Next, we perform the Cholesky decomposition of Ω (choosing lower triangular specification) to finally arrive at Σ, which follows the approach in Ang and Piazzesi (2003). Finally, since the short rate is modelled as part of the state vector, the vector (δ_0, δ_1) corresponds to the last point and row of μ and Φ, respectively, and is estimated during the first step as well.

In the second step, taking the estimates of (μ, Φ, Σ) as given, we estimate the risk premia parameters (λ_0, λ_1) which correspond to the evolution of the state vector risk prices and which are linking physical probability measure P with the risk-neutral probability measure Q via the Radon-Nykodim density process (9). We estimate (λ_0, λ_1) by minimizing the sum of squared fitting errors of the model equity and yields. Therefore, given that model implied yields satisfy equation (26) and that the model implied price to dividend ratios satisfy equation (37), the optimal risk premia parameters are chosen to minimize[13]

$$MSSE = \sum_{t=1}^{T} \sum_{n=1}^{N} \frac{(\hat{y}_t^n - y_t^n)^2}{TN} + \sum_{t=1}^{T} \frac{\left(\frac{\hat{V}_t}{\hat{D}_t} - \frac{V_t}{D_t} \right)^2}{T} \tag{38}$$

where N is the number of maturities available for rates, $\hat{y}_t^n$ and y_t^n denote the model and the market yields of maturity n, respectively, and $\frac{\hat{V}_t}{\hat{D}_t}$ and $\frac{V_t}{D_t}$ denote the model and the market price to dividend ratios.[14] This approach is employed, for instance, in Mönch (2008b), Cochrane and Piazzesi (2009), Ang and Ulrich (2012), Jardet et al. (2013). Albeit this method is less efficient than a one-step maximum likelihood estimation of the joint distribution of the state vector, it is much faster and yield unbiased estimates[15].

We impose the restrictions sufficient to guarantee that the prices of dividend strips converge to zero as their maturities approach infinity. Such condition on the limiting behavior of dividend strips is necessary to justify the truncation of the sum in equation (37). Empirically, the sum has been truncated at 20 years. These restrictions are non-linear in risk premia parameters and also involve constraints on the absolute values of eigenvalues Φ^Q.[16] As a result, the optimization problem (38) is highly non-linear in the estimated parameters and is subject to non-linear constraints on the estimated parameters.

In this highly non-linear system, finding appropriate starting value for the minimization problem is crucial to achieving convergence. We use several estimation steps which to some extent mimic the iterative procedure employed in Ang and Piazzesi (2003) and Mönch (2008a) and which generally appears in the affine term structure model literature. We proceed in the following way. We start the

[13] This type of quadratic criterion is equivalent to assuming that market quotes are a mixture between the "true" quote and an Gaussian disturbance, yielding unbiased estimates.

[14] We acknowledge the fact that yields and price to dividend ratios have different scalings. However, we employ interior point method to minimize the objective function (38) which uses gradient methods and is relatively robust to variables scaling.

[15] Ang and Piazzesi (2003) reports that single step maximum likelihood optimization typically produce unacceptable yields dynamics. Mönch (2008b) also reports that such objective specification is better suited the recursive out-of-sample forecast routines.

[16] Despite imposing highly non-linear restriction on the parameters, in general we find that they actually improve the estimation procedure by promptly rejecting parameter regions where dividend strips diverge in the limit and thus where parameters thus do not solve the minimization problem (38).

estimation by assuming that risk premia are non-zero and constant by setting λ_1 to zero matrix, and allowing λ_0 to vary. We first minimize the sum of squared errors in equation (38) with respect to the rates term structure. Once λ_0 is estimated, in the next estimation step we let the risk premia to be time-variant (i.e. we let both λ_0 and λ_1 vary) and minimize the same sum of squared errors as in previous step. Once the risk premia are estimated, we generate random matrices around this point and re-run the estimation using them as new starting values.[17][18] We then select the estimators that achieve lowest value for (38). Next, we once again generate random matrices around these estimators and pass them as starting values to the joint estimation. While estimating risk premia parameters to the yields term structure, we also impose the constraint that guarantees that prices of dividend strips are converging to zero (Appendix C). At the same time, this guarantees that the initial point for joint estimation is admissible. Finally, we run joint estimation, where we minimize the MSSE for both the term structure of rates and price to dividend ratios. We select the risk premia estimates that minimize the criterion presented in equation (38). Standard errors and corresponding p-values of the risk premium parameters are then computed using the numerical Hessian of the sum of squared fitting errors (38) at the final point. We perform this procedure for each economy (US, Europe) and each equity index (S&P500, Dow Jones, Russell, MSCI Europe, MSCI Germany, MSCI France) separately, using the term structure of their corresponding economy.

4.3 Parameter estimation

Table 3 presents the estimates obtained when estimating the VAR model with one lag. For both the US and European datasets, the short rate dynamics is the most persistent, displaying a diagonal element in the Φ matrix that is above 0.95. Then Factor 1 and 3 are found to be more persistent that Factor 2. Most of the parameters in Φ and μ are statistically different from zero at a 5% risk level. Note that standard information criteria would point towards a VAR with a higher number of lags, as indicated in the Table 4. However, it would imply a much larger number of parameters to be calibrated later: each Φ matrix incorporates 16 parameters. With one lag, the model already requires 20 parameters to be calibrated. With two lags, the number of parameters to be estimated would increase to 36, which would complicate the estimation given the 105 quarters of available data.

Table 3: US & Europe state vector VAR(1) dynamics estimates

	US					Europe				
	μ	Φ				μ	Φ			
F1	-0.54	0.87	-0.46	0.39	0.22	-0.26	0.88	-0.47	0.38	0.19
	(0.2)	(0)	(0)	(0)	(0.07)	(0.54)	(0)	(0)	(0)	(0.21)
F2	-0.79	0.22	0.84	0.22	0.28	-0.67	0.24	0.82	0.19	0.21
	(0.02)	(0)	(0)	(0)	(0)	(0.04)	(0)	(0)	(0)	(0.07)
F3	0.05	-0.15	0.00	0.86	-0.01	0.12	-0.16	0.00	0.88	-0.02
	(0.84)	(0)	(0.96)	(0)	(0.85)	(0.65)	(0)	(0.96)	(0)	(0.83)
r_t	0.02	0.01	-0.03	0.00	0.97	-0.01	0.02	-0.01	0.00	0.96
	(0.79)	(0.01)	(0)	(0.68)	(0)	(0.87)	(0)	(0.03)	(0.87)	(0)

The table provides multivariate OLS estimates of the state vector VAR(1) P-measure dynamics $X_t = \mu + \Phi X_{t-1} + \Sigma\varepsilon_t$, where $X_t = [F1, F2, F3, r_t]'$. Factors (F1, F2, F3) used in the estimation are the same for US and Europe. Short rate r_t is 3-month US government yield in case of US, and 3 month Germany government yield in case of Europe. Associated p-values are reported in round brackets.

Table 5 presents estimates of the risk premium parameters in case of the US economy. A first attempt to calibrate a unique pricing kernel for all three equity indices in the meantime failed at converging

[17]The need for this approach is in part motivated by high sensitivity of to the initial point specification, even if following steps are equal.

[18]Ideally, we would shift the matrix of risk premia parameters in all possible directions to generate a full set of initial values and then run the estimation. However, due to high number of parameters estimated (20), this approach is infeasible. Generating random matrices around the admissible point adheres to the same idea but reduces number of directions initiated.

Table 4: Information criterias for the historical VAR model.

	United States			Europe		
Information Criterias	1 lag	2 lags	3 lags	1 lag	2 lags	3 lags
AIC	1357.07	1300.89	1277.87	1271.57	1222.33	1211.06
BIC	1389.14	1356.82	1357.49	1325.03	1318.21	1349.06
HQ	1370.07	1323.56	1310.13	1293.24	1261.19	1266.98

The table provides AIC, BIC, HQ information criteria estimates of the loglikelihood function associated with Gaussian VAR(p) P-measure dynamics of the state vector $X_t = \mu + \Phi(p)X_{t-1} + \Sigma\varepsilon_t$, where $p = 1, 2, 3$ and $X_t = [F1, F2, F3, r_t]'$. Factors (F1, F2, F3) used in the estimation are the same for US and Europe. Short rate r_t is 3-month US government yield in case of US, and 3 month Germany government yield in case of Europe.

towards a decent solution. The table presents the estimates obtained when calibrating the risk premia for each given equity index at a time. This calibration is obtained by minimizing numerically the sum of mean square errors between the market price to dividend ratio and its model implied counterpart, and market and model implied yields. Equivalent results in the case of Europe are presented in Table 6. With the exception of the MSCI France index, mainly only the parameters belonging to the drift λ_0 are statistically different from zero: our results indicate that the change in measure impacts mainly the long run drift of the dynamics under the risk neutral distribution. Interactions between macroeconomic variables under the historical and risk neutral distributions are found to be very similar.

Table 5: US risk premia estimates

	S&P500					Dow Jones					Russell				
	λ_0	λ_1				λ_0	λ_1				λ_0	λ_1			
F1	-11.80	0.10	0.18	1.19	-2.44	15.24	-1.91	0.99	0.76	-4.46	16.38	1.54	-0.57	2.03	-5.24
	(0)	(0.74)	(0.4)	(0.05)	(0.13)	(0)	(0.21)	(0.35)	(0.76)	(0)	(0)	(0.62)	(0.89)	(0.73)	(0)
F2	-50.40	3.98	1.80	2.93	-18.58	-93.03	-5.62	3.52	2.13	-11.22	-278.70	0.43	0.58	0.51	3.16
	(0)	(0)	(0)	(0)	(0)	(0)	(0.13)	(0.05)	(0.15)	(0)	(0)	(0.87)	(0.93)	(0.86)	(0.05)
F3	-201.50	-2.63	-0.88	-1.63	11.41	-115.99	10.11	-4.72	-0.47	17.82	-147.51	-3.57	1.29	-4.34	17.27
	(0)	(0)	(0.01)	(0)	(0)	(0)	(0)	(0.09)	(0.78)	(0)	(0)	(0.37)	(0.77)	(0.43)	(0)
r_t	-94.07	2.20	1.06	1.83	-7.33	-30.13	-2.55	1.50	0.75	0.74	-113.82	0.09	0.23	0.00	7.45
	(0)	(0)	(0)	(0)	(0)	(0)	(0.53)	(0.52)	(0.6)	(0.39)	(0)	(0.95)	(0.91)	(1)	(0)

The table reports estimates of the essentially affine risk premium specification $\lambda_t = \lambda_0 + \lambda_1 X_t$. Reported λ_0, λ_1 minimize the objective function (38). Risk prices are calibrated per each US equity index individually, using the corresponding domestic rates term structure. Associated p-values are reported in round brackets and are computed using numerical Hessian of the objective function at the solution point.

Table 6: Europe risk premia estimates

	MSCI Europe					MSCI Germany					MSCI France				
	λ_0	λ_1				λ_0	λ_1				λ_0	λ_1			
F1	-27.57	-0.91	-0.80	0.39	12.91	84.07	0.60	-0.19	2.21	4.50	138.29	2.27	-0.88	3.78	10.85
	(0)	(0.89)	(0.68)	(0.93)	(0)	(0)	(0.99)	(1)	(0.94)	(0.84)	(0)	(0)	(0)	(0)	(0)
F2	-104.59	2.19	1.47	-0.31	-34.47	-21.92	0.80	0.25	3.77	7.76	25.14	1.67	0.04	3.70	-16.32
	(0)	(0.28)	(0.51)	(0.86)	(0)	(0)	(0.96)	(0.99)	(0.96)	(0.79)	(0)	(0)	(0.9)	(0)	(0)
F3	-248.04	-10.48	-1.46	3.69	-11.82	-89.48	1.82	-1.55	-0.06	10.08	-126.36	1.69	-3.37	-2.62	12.64
	(0)	(0)	(0.45)	(0.06)	(0.41)	(0)	(0.91)	(0)	(1)	(0.78)	(0)	(0)	(0)	(0)	(0)
r_t	30.95	2.44	0.75	-0.78	-7.59	-52.81	-0.85	0.68	-0.45	-1.07	-61.85	-1.65	1.62	-0.73	-17.42
	(0)	(0.01)	(0.94)	(0.9)	(0.56)	(0)	(0.94)	(0.98)	(0.99)	(0.99)	(0)	(0)	(0)	(0)	(0)

The table reports estimates of the essentially affine risk premium specification $\lambda_t = \lambda_0 + \lambda_1 X_t$. Reported λ_0, λ_1 minimize the objective function (38). Risk prices are calibrated per each European equity index individually, using the corresponding domestic rates term structure. Associated p-values are reported in round brackets and are computed using numerical Hessian of the objective function at the solution point.

4.4 Bubble diagnosis

Our bubble diagnosis mechanics works as follows. We will refer to V_t as the market price for a stock index and V_t^* as its model implied counterpart obtained from the previous estimations. The deviation of market price from its fundamental counterpart is thus measured as $z_t = \frac{V_t}{V_t^*} - 1$. A market is diagnosed to have been driven by a bubble when z_t is non-stationary. Stationarity of z_t implies that the price of a given stock index cannot stay long away from where its fundamentals are. A bubble occurs when precisely the opposite is observed: the index trades at a level that remains far from its fundamental valuation for an extended period of time.

Figure 4 compares V_t and V_t^* for each equity index. Consistently with results displayed in Figure 3, the S&P500 and the Dow Jones are showing a significant discrepancy to their fundamental valuation around 2001. The ratio z_t is charted in Figure 5: in 2001, the equity over-valuation compared to the model-implied fundamental valuation reached about 60% in the case of the S&P500 and 50% in the case of the Dow Jones. On the contrary, with the exception of the MSCI France index, discrepancies of the same magnitude are not found in the rest of the indices.

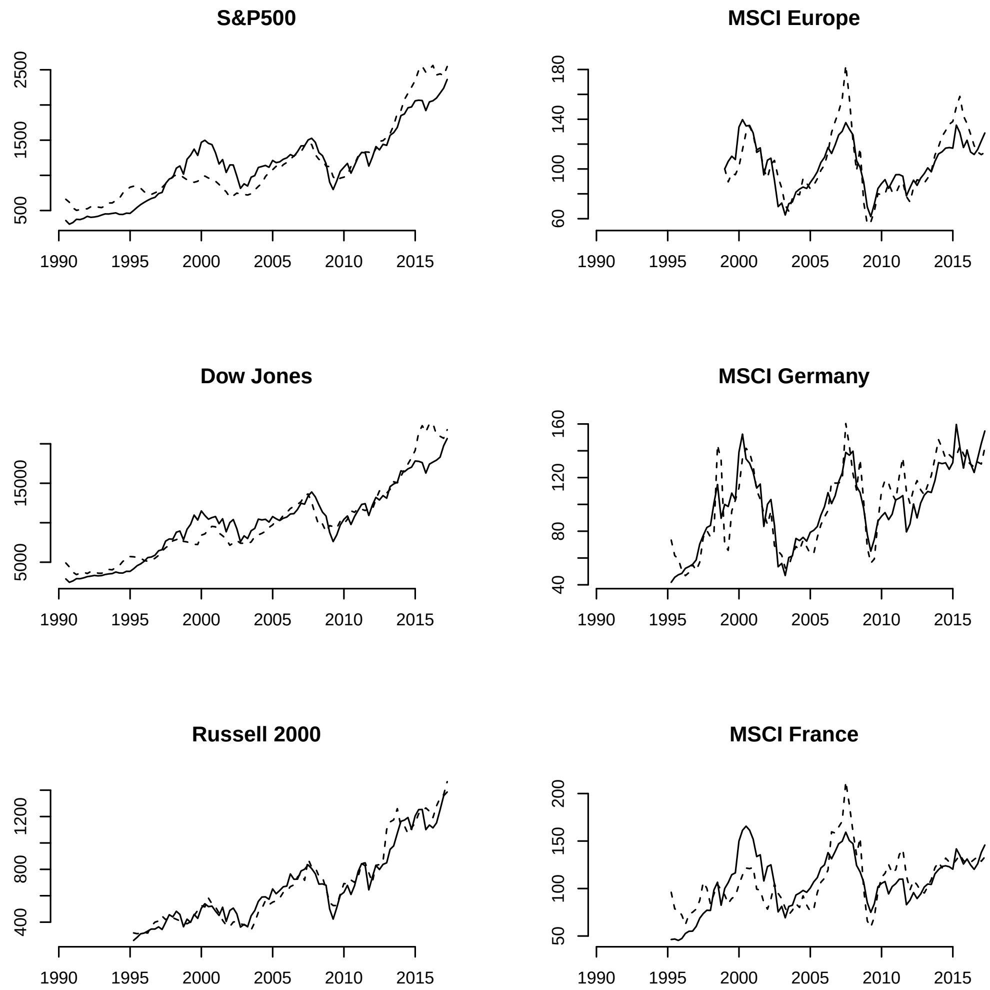

Figure 4: Time-series evolution of market (bold line) and model (dashed line) equity prices in US and Europe over the 1990-2017 period.

Graphical analysis alone is not sufficient to statistically diagnose a bubble. Inspired by the previously mentioned literate, our test methodology will be based on the analysis of tests for non-stationarity. Here, we rely on the Augmented Dickey-Fuller (ADF) and Philip and Perron unit root tests for stationarity. For each z_t time series, Table 7 presents results of this test. Only the S&P500, the Dow Jones and the MSCI France have been found to display a non-stationary z_t: we can suspect that for these indices, a portion of their price history has been driven by a bubble. Interestingly, when comparing these results to the outcome of such a test for the price-to-dividend ratios, a similar conclusion

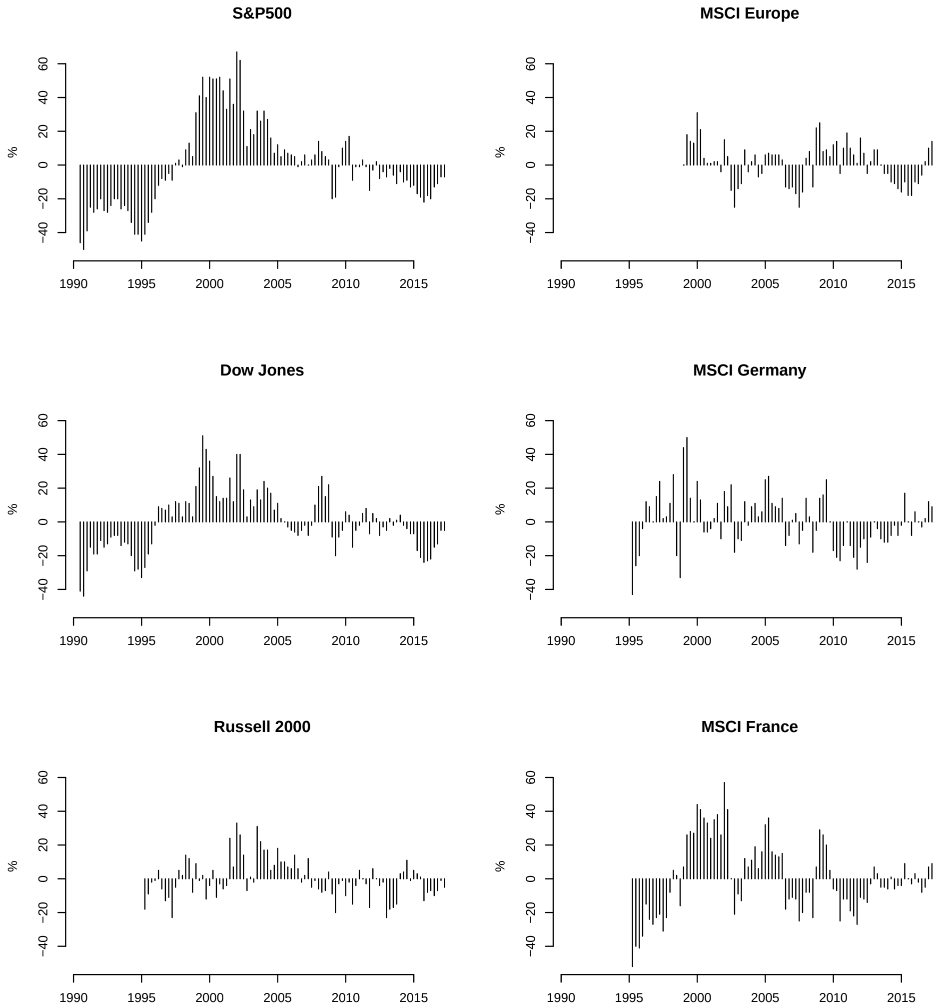

Figure 5: Time-series evolution of the valuation measure over the 1990-2017 period.

can be reached – but in the case of MSCI Europe for which the price-to-dividend ratio is found to be stationary, when the attached z_t are not. A further analysis is necessary to discriminate between the price-to-dividend ratio and the z_t as a bubble measure: that is what is going to be detailed now.

As pointed out in Gürkaynak (2008), using an approach that compares a model-based valuation to market prices and finding a non-stationary discrepancy between both is not enough to diagnose a bubble: this discrepancy could have also been the result of model mis-specification. In order to increase the level of certainty with which we indicate the presence of bubbles, we rely on a second element: an analysis of the consequences of z_t over its underlying economy. Also important to note is the fact that the model is applied to six equity indices and the goodness of fit for a majority of them is a first sign that model mis-specification risk is somewhat limited: 3 out of the 6 indices exhibit discrepancies z_t that are stationary.

A significant attention in the economic literature has been given to the potential impacts of a bubble on an economy, both over its expansion and deflation periods. This literature focused on the impact of bubbles on both the real economy and financial markets. When it comes to the economy, both Martin and Ventura (2012) and Hirano and Yanagawa (2016) ackowledge the fact that the beginning

Table 7: Philip and Perron tests for unit roots

		Stationarity test					
Test type		S&P500	Dow Jones	Russell 2000	MSCI Europe	MSCI Germany	MSCI France
PP	Price Dividend Ratio	0.65	0.38	0.02	0.49	0.01	0.27
	Spread to macro model	0.56	0.10	0.01	0.02	0.01	0.12

The table reports the p-value obtained for the Philip and Perron unit root test applied to discrepancies between market and fundamental equity prices z_t and to the price-to-dividend ratios. The null hypothesis of the test is the presence of the unit root, which implies non-stationarity time series.

and the end of bubble have different impact on an economy: the ignition of a bubble has a tendency to lead to an increase in the aggregate revenue of an economy and to the increase in the existing stock of capital fuelling investment. The burst of a bubble has the opposite effects, in some cases leading to a permanently lower level of investment, as discussed in Hirano and Yanagawa (2016) (in Section 6). On the financial market side, the collapse of a bubble has been recognized as the source of an increased volatility (see Simon (2003)) and as having adverse consequences on liquidity (see Brunnermeier (2010)). Liquidity gets scarce when the bubble collapses, but then liquidity injections by Central Banks help mitigate this effect. In order to assess whether a non-stationary of z_t could indicate a bubble, we perform the estimation of a VAR model that combines economic data and and z_t, and observe the impact of a shock in z_t on the underlying economy through an impulse-response function (IRF hereafter).

We estimate the following VAR(p) model:

$$Y_t = (z_t\ F_{1,t}\ F_{2,t}\ F_{3,t}\ GDP_t\ Inv_t\ Liq_t\ VIX_t)\,, \tag{39}$$

$$Y_t = \Psi_0 \sum_{i=1}^{p} \Psi_i Y_{t-i} + \nu_t, \tag{40}$$

where $F_{i,t}$ is the i^{th} macroeconomic factor from Section 2 used here a control variables[19], GDP_t is the quarterly variation in the local economy's real GDP, Inv_t is the same variation in its investments, Liq_t is a measure of markets' liquidity and VIX_t is the value of the VIX index at time t. Liquidity is measured as the different between 3-month libor rate and its equivalent for government bonds, representing the funding cost for banks.

The optimal lag is obtained using the Hannan and Quin (HQ) information criterion. Next, an impulse response function is derived from those estimations. Results for the IRFs are presented in Figure 6 for the S&P500, Figure 8 for the Dow Jones and Figure 9 for the Russell 2000. IRFs seem to confirm that a bubble has been present in the S&P500 and Dow Jones, but not in the Russell 2000 index: both for the S&P500 and the Dow Jones indices, a shock to z_t leads to an increase in the output and investment in the short term and induces a long lasting decrease in them in the longer term. Besides, the burst of the bubble leads to a statistically short term increase in volatility. Liquidity is also improving at the beginning of a bubble (as captured by a lower difference between Libor and government rates), and then substantially deteriorates due to the collapse of the bubble throughout a number of quarters. When looking at the Russell 2000 results, none of these elements are found to be statistically significant: z_t does not lead to significant changes in the four economic variables. As a sanity check, we also produce results for the MSCI Europe, adapting the control variables to the Eurozone[20]. Similar to Russell 2000, we find no sign that the discrepancies z_t attached to the MSCI Europe index have led to the expected consequences of a bubble (Figure 10).

Finally, a comparison can be made with price-to-dividend ratios: as showed in Table 7, a stationarity test applied to the price-to-dividend ratio and to the spreads between market prices and the model-implied corresponding value would yield very similar results, but in the case of the MSCI France. Non stationarity is a symptom of a bubble behavior, but not only: a change in regime would suffice to create non stationarity. Missing a stationarity test cannot be interpreted as being driven by a bubble.

[19] Consistently with the previously mentioned FAVAR literature.

[20] We refrain from doing the same with the MSCI Germany and France as their interaction with the overall European economy makes the control variable selection more complex.

When running the IRF as presented earlier but replacing the z_t with the price-to-dividend ratio yield Figure 7in the case of the S&P500[21]. Even if an elevation of the price-to-dividend ratio leads to an increased economic activity and investment, the longer lasting consequences of a bubble are nowhere to be found: an increase in the ratio does not lead to a mechanical value destruction in the economy as found when investigating a shock in z_t. The z_t appear as having a interesting marginal contribution over the price-to-dividend ratio in this respect.

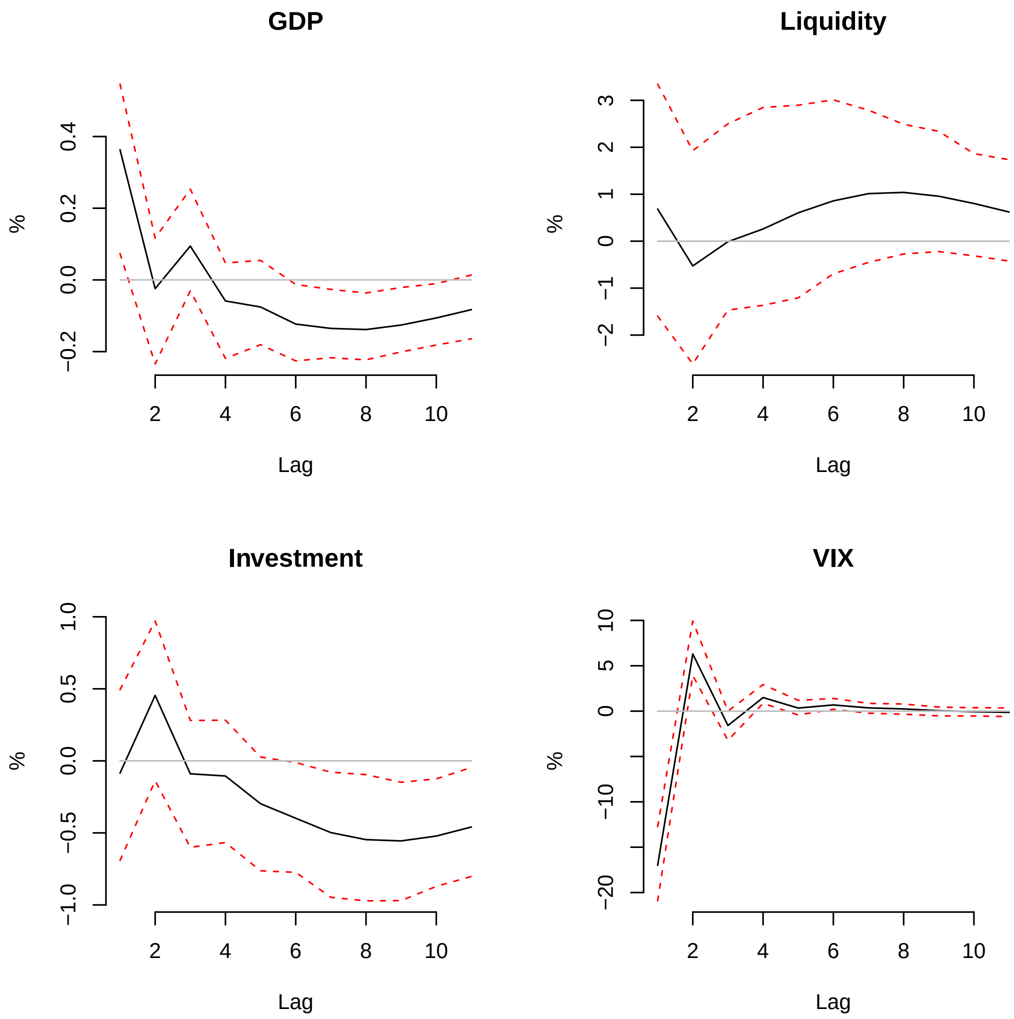

Figure 6: Impulse responses of US GDP, Liquidity, Investment and VIX to the bubble in S&P500 equity index.

5 Conclusion

We propose a new approach to test for bubbles in equity prices. Bubbles are defined as a long lasting departure between a fundamental valuation of equities and their market price. Fundamental valuation is obtained by from an affine model that relates the dividend growth rate to macro-economic factors. The estimation of the model provides a fundamental price for equity indices. The bubble diagnosis

[21] Similar results have been found with the different equity indices used in this article.

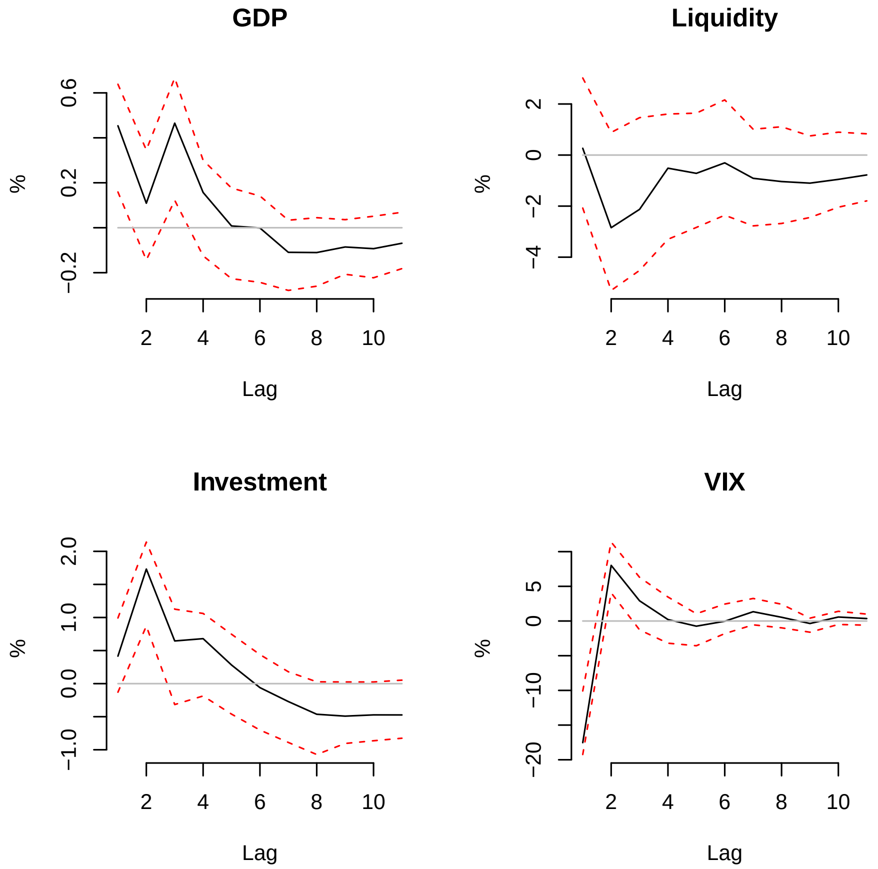

Figure 7: Impulse responses of US GDP, Liquidity, Investment and VIX to a shock in the price to dividend ratio

arise from the analysis of the discrepancy between market and model prices: a bubble leads to a non-stationary discrepancy that weigh the corresponding economy by impacting its output and investment. Using our approach, we diagnose bubbles both for the S&P500 and the Dow Jones over the 1990-2017 period but not for the four other equity indices for which we ran the test. The economic foundations of the affine model used here pledge for a limited model mis-specification risk, a usual caveat for bubble indicators.

Appendices

A Derivation of bonds recursive ODE equations

In this section, we derive a system of ODE equations that arises when pricing bonds in the ATS framework. Assuming that the price of a bond maturing in n years at time t, p_t^n is exponential affine in the state vector

$$p_t^n = \exp\left(A_n + B_n' X_t\right) \tag{41}$$

we show that the price of of the bond maturity in $n+1$ years is also exponential affine as follows

$$\begin{aligned}
p_t^{n+1} &= E_t\left[m_{t+1} p_{t+1}^n\right] \\
&= E_t\left[\exp\left(-r_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1} + A_n + B_n' X_{t+1}\right)\right] \\
&= \exp\left[-r_t - \frac{1}{2}\lambda_t'\lambda_t + A_n\right] E_t\left[\exp\left(-\lambda_t'\varepsilon_{t+1} + B_n' X_{t+1}\right)\right] \\
&= \exp\left[-r_t - \frac{1}{2}\lambda_t'\lambda_t + A_n\right] E_t\left[\exp\left(-\lambda_t'\varepsilon_{t+1} + B_n'\left(\mu + \Phi X_t + \Sigma\varepsilon_{t+1}\right)\right)\right] \\
&= \exp\left[-\delta_0 + A_n + B_n'\mu + \left(B_n'\Phi - \delta_1'\right) X_t - \frac{1}{2}\lambda_t'\lambda_t\right] \times E_t\left[\exp\left(\left(-\lambda_t' + B_n'\Sigma\right)\varepsilon_{t+1}\right)\right] \\
&= \exp\left[-\delta_0 + A_n + B_n'\mu + \left(B_n'\Phi - \delta_1'\right) X_t - \frac{1}{2}\lambda_t'\lambda_t + \frac{1}{2}\left(-\lambda_t' + B_n'\Sigma\right)\left(-\lambda_t' + B_n'\Sigma\right)'\right] \\
&= \exp\left[-\delta_0 + A_n + B_n'\mu + \left(B_n'\Phi - \delta_1'\right) X_t + \frac{1}{2}B_n'\Sigma\Sigma' B_n - B_n'\Sigma\lambda_t\right] \\
&= \exp\left[-\delta_0 + A_n + B_n'\mu + \left(B_n'\Phi - \delta_1'\right) X_t + \frac{1}{2}B_n'\Sigma\Sigma' B_n - B_n'\Sigma\left(\lambda_0 + \lambda_1 X_t\right)\right] \\
&= \exp\left[-\delta_0 + A_n + B_n'\left(\mu - \Sigma\lambda_0\right) + B_n'\left(\Phi - \Sigma\lambda_1\right) X_t - \delta_1' X_t + \frac{1}{2}B_n'\Sigma\Sigma' B_n\right] \\
&= \exp\left[-\delta_0 + A_n + B_n'\mu^Q + B_n'\Phi^Q X_t - \delta_1' X_t + \frac{1}{2}B_n'\Sigma\Sigma' B_n\right] = \exp\left(A_{n+1} + B_{n+1}' X_t\right)
\end{aligned} \tag{42}$$

Matching the coefficients on the RHS and LHS of equation (42) leads to the system of ODE equations

$$A_n = A_{n-1} + B_{n-1}'(\mu - \Sigma\lambda_0) + \frac{1}{2}B_{n-1}'\Sigma\Sigma' B_{n-1} - \delta_0 \tag{43}$$

$$B_n' = B_{n-1}'(\Phi - \Sigma\lambda_1) - \delta_1' \tag{44}$$

where we used a general specification of the short rate $r_t = \delta_0 + \delta_1' X_t$. The boundary conditions are given by

$$A_1 = -\delta_0 \tag{45}$$

$$B_1' = -\delta_1' \tag{46}$$

Finally, short rate specification used throughout this work corresponds to the case when

$$\delta_0 = 0 \tag{47}$$

$$\delta_1 = \delta^r \tag{48}$$

B Derivation of equity recursive ODE equations

In this section, we derive a system of ODE equations, arising when pricing price to dividend ratio (or normalized dividend strips) under risk-neutral probability measure Q.

The price of a single dividend strip that pays at time $t+\tau$ is given by equation (32), which we reiterate here for convenience

$$V_{t,n}^{d}=E_{t}^{Q}\left[e^{-\sum_{k=1}^{n}r_{t+k-1}}\frac{D_{t+n}}{D_{t}}\right]=E_{t}^{Q}\left[\exp\left(\sum_{k=1}^{n}d_{t+k}-r_{t+k-1}\right)\right] \tag{49}$$

The derivation starts with the guess that price of dividend strip is exponential affine in the state vector X_t

$$V_{t,n}^{d}=\exp\left(a_{n}+b_{n}^{\prime}X_{t}\right) \tag{50}$$

where X_t follows Gaussian VAR as given by equation (5). First, we notice that we can re-write using the guess given by equation (50), the price of dividend strip recursively as follows

$$\begin{aligned}V_{t,n}^{d}&=E_{t}^{Q}\left[\exp\left(\sum_{k=1}^{n}d_{t+k}-r_{t+k-1}\right)\right]=E_{t}^{Q}\left[\exp\left(d_{t+1}-r_{t}\right)\exp\left(\sum_{k=2}^{n}d_{t+k}-r_{t+k-1}\right)\right]\\&=E_{t}^{Q}\left[\exp\left(d_{t+1}-r_{t}\right)E_{t+1}^{Q}\left[\exp\left(\sum_{k=2}^{n}d_{t+k}-r_{t+k-1}\right)\right]\right]=E_{t}^{Q}\left[\exp\left(d_{t+1}-r_{t}\right)V_{t+1,n-1}^{d}\right]\\&=E_{t}^{Q}\left[\exp\left(d_{t+1}-r_{t}\right)\exp\left(a_{n-1}+b_{n-1}^{\prime}X_{t+1}\right)\right]\end{aligned} \tag{51}$$

where we used the law of iterated expectations in the third equality and equation (50) in the last equality.
We assume that earnings logarithmic growth rates are affine in the state vector X_t (which implies that earnings themselves are exponential affine in the state vector X_t)

$$g_{t}=\gamma_{0}+\gamma_{1}^{\prime}X_{t} \tag{52}$$

where γ_0, γ_1 are constant parameters. Next, we assume that logarithmic growth rates of the payout ratio are also affine in the state vector X_t

$$c_{t}=\alpha_{0}+\alpha_{1}g_{t} \tag{53}$$

This implies that dividend (log) growth rates d_t satisfy

$$d_{t}=\ln\frac{D_{t}}{D_{t-1}}=\ln\frac{E_{t}C_{t}}{E_{t-1}C_{t-1}}=c_{t}+g_{t}=\alpha_{0}+\left(1+\alpha_{1}\right)\gamma_{0}+\left(1+\alpha_{1}\right)\gamma_{1}^{\prime}X_{t}=:\omega_{0}+\omega_{1}^{\prime}X_{t} \tag{54}$$

where $\omega_0=\alpha_0+(1+\alpha_1)\gamma_0$ and $\omega_1=(1+\alpha_1)\gamma_1$ are obtained from earnings and payout ratio growths loadings on the state vector factors. d_t is thus also affine in X_t.

Having dividend growth rate process in place, we can further rewrite equation (51) as follows (directly under Q, using the same pricing kernel as for bonds pricing)

$$\begin{aligned}V_{t,n}^{d}&=E_{t}^{Q}\left[\exp\left(d_{t+1}-r_{t}\right)\exp\left(a_{n-1}+b_{n-1}^{\prime}X_{t+1}\right)\right]\\&=E_{t}^{Q}\left[\exp\left(\omega_{0}+\omega_{1}^{\prime}X_{t+1}-r_{t}\right)\exp\left(a_{n-1}+b_{n-1}^{\prime}X_{t+1}\right)\right]\\&=\underbrace{\exp\left(\omega_{0}-\delta_{0}-\delta_{1}^{\prime}X_{t}+a_{n-1}\right)}_{C}E_{t}^{Q}\left[\exp\left(\left(\omega_{1}^{\prime}+b_{n-1}^{\prime}\right)X_{t+1}\right)\right]\\&=C\cdot\exp\left[\left(\omega_{1}^{\prime}+b_{n-1}^{\prime}\right)\left(\mu^{Q}+\Phi^{Q}X_{t}\right)+\frac{1}{2}\left(\omega_{1}^{\prime}+b_{n-1}^{\prime}\right)\Sigma\Sigma^{\prime}\left(\omega_{1}+b_{n-1}\right)\right]\\&=\exp\left(a_{n}+b_{n}^{\prime}X_{t}\right)\end{aligned} \tag{55}$$

where we used the guess from equation (50) in the last equality and the fact that state vector follows conditional Gaussian VAR $X_{t+1}=\mu^Q+\Phi^Q X_t+\Sigma\varepsilon_{t+1}$ under Q (under essentially affine risk premia specification used throughout this work).
Similar to bonds pricing, in order for the last equality to hold as an identity, coefficients a_n and b_n

have to satisfy the following system of equations (obtained via coefficients matching in equation (55))

$$
\begin{aligned}
a_n &= a_{n-1} + \omega_0 + (\omega_1 + b_{n-1})' \mu^Q + \frac{1}{2} (\omega_1 + b_{n-1})' \Sigma\Sigma' (\omega_1 + b_{n-1}) - \delta_0 && (56)\\
b_n &= \Phi^{Q'} (\omega_1 + b_{n-1}) - \delta_1 && (57)
\end{aligned}
$$

To obtain boundary conditions for a_n and b_n, we note that for 1-period ahead dividend strip, we have (again, using equation (50))

$$
\begin{aligned}
V^d_{t,1} &= E^Q_t [\exp(d_{t+1} - r_t)] = E^Q_t [\exp(\omega_0 + \omega_1' X_{t+1} - \delta_0 - \delta_1' X_t)] = \\
&= \exp(\omega_0 - \delta_0 - \delta_1' X_t) E^Q_t [\exp(\omega_1' X_{t+1})] \\
&= \exp\left(\omega_0 - \delta_0 - \delta_1' X_t + \omega_1' \left(\mu^Q + \Phi^Q X_t\right) + \frac{1}{2}\omega_1' \Sigma\Sigma' \omega_1\right) \\
&= \exp(a_1 + b_1' X_t)
\end{aligned} \tag{58}
$$

as previously, last equality must hold as an identity. Matching the coefficients in equation (58), we obtain the following boundary conditions

$$
\begin{aligned}
a_1 &= -\delta_0 + \omega_0 + \omega_1' \mu^Q + \frac{1}{2}\omega_1' \Sigma\Sigma' \omega_1 && (59)\\
b_1 &= -\delta_1 + \Phi^{Q'} \omega_1 && (60)
\end{aligned}
$$

The short rate specification used throughout this work corresponds to the case when

$$
\begin{aligned}
\delta_0 &= 0 && (61)\\
\delta_1 &= \delta^r && (62)
\end{aligned}
$$

C Parameter closed-form solutions and limiting behaviour

In this section we analyze the system of recursive ODE equations (56), (57) derived in Appendix B. equation (57) implies that b_n follows a simple recursive equation with initial boundary condition given by equation (60) and is independent of a_n. Using the boundary condition, we can rewrite equation (57) as follows (we temporarily drop the superscript Q' from $\Phi^{Q'}$ to ease the notations)

$$
\begin{aligned}
b_n &= \Phi(\omega_1 + b_{n-1}) - \delta_1 = \Phi b_{n-1} + \Phi\omega_1 - \delta_1 \\
&= \Phi[\Phi b_{n-2} + \Phi\omega_1 - \delta_1] + \omega_1 - \delta_1 = \dots \\
\dots &= \Phi^{n-1} b_1 + \sum_{k=1}^{n-1} \Phi^k \omega_1 - \sum_{k=0}^{n-2} \Phi^k \delta_1 \\
&= \sum_{k=1}^{n} \Phi^k \omega_1 - \sum_{k=0}^{n-1} \Phi^k \delta_1
\end{aligned} \tag{63}
$$

where by convention we set $\Phi^0 = I$. Since the terms δ_1 and ω_1 are constant, last equation of equation (63) represents a finite sum of a geometric series, which admits a closed form solution and is given by

$$
\begin{aligned}
b_n &= \sum_{k=1}^{n} \Phi^k \omega_1 - \sum_{k=0}^{n-1} \Phi^k \delta_1 \\
&= \Phi(I - \Phi^n)(I - \Phi)^{-1} \omega_1 - (I - \Phi^n)(I - \Phi)^{-1} \delta_1
\end{aligned} \tag{64}
$$

In equation (64) we require that matrix $I - \Phi$ is invertible. This condition is equivalent to absolute value of eigenvectors of matrix Φ being different from one. This condition alone doesn't guarantee the convergence of Φ^n (and hence b_n). However, as long as absolute values of eigenvalues of matrix Φ are also less than one, we have

$$\lim_{n\to\infty} \Phi^n = 0 \tag{65}$$

which implies $I - \Phi$ is invertible and that in the limit $n \to \infty$ parameter b_n satisfies

$$b_\infty = \Phi \left(I - \Phi\right)^{-1} \omega_1 - \left(I - \Phi\right)^{-1} \delta_1 \tag{66}$$

and is also known as the fixed point of equation (57).[22] Importantly, restriction on eigenvalues is not the only restriction that can be imposed on matrix Φ that leads to stable results for b_n. These restriction present sufficient, but not necessary condition for b_n convergence.

Next, from equation (56), we have

$$\begin{aligned} a_n &= a_{n-1} + \omega_0 + \left(\omega_1 + b_{n-1}\right)' \mu^Q + \frac{1}{2}\left(\omega_1 + b_{n-1}\right)' \Sigma\Sigma' \left(\omega_1 + b_{n-1}\right) - \delta_0 \\ &= a_1 + (n-1)\,\omega_0 + (n-1)\,\omega_1'\mu^Q + \left(\sum_{k=1}^{n-1} b_k'\right)\mu^Q + (n-1)\,\frac{1}{2}\omega_1'\Sigma\Sigma'\omega_1 + \ldots \\ \ldots &+ \omega_1'\Sigma\Sigma'\left(\sum_{k=1}^{n-1} b_k\right) + \frac{1}{2}\sum_{k=1}^{n-1} b_k'\Sigma\Sigma' b_k - (n-1)\,\delta_0 \end{aligned} \tag{67}$$

equation (67) provides an analytical expression for a_n in terms of model parameters, since the solution for b_n was derived in equation (64) and boundary condition a_1 is given in equation (59).
As evident from equation (67), in order to analyze the limiting behaviour of a_n, we first have to analyze the limiting behaviour of $\sum_{k=1}^{n-1} b_k$. From equation (64), we have

$$\begin{aligned} \sum_{k=1}^{n} b_k &= \sum_{k=1}^{n} \Phi\left(I - \Phi^k\right)\left(I - \Phi\right)^{-1}\omega_1 - \sum_{k=1}^{n}\left(I - \Phi^k\right)\left(I - \Phi\right)^{-1}\delta_1 \\ &= n\Phi\left(I - \Phi\right)^{-1}\omega_1 - \Phi^2\left(I - \Phi^n\right)\left(I - \Phi\right)^{-2}\omega_1 - \ldots \\ &\ldots - n\left(I - \Phi\right)^{-1}\delta_1 + \Phi\left(I - \Phi^n\right)\left(I - \Phi\right)^{-2}\delta_1 \end{aligned} \tag{68}$$

where, as previously, we assume that matrix $I - \Phi$ is invertible. Note that under the restriction that absolute values of eigenvalues of Φ are less than one, using equation (65) in the limit we have

$$\begin{aligned} \lim_{n\to\infty}\sum_{k=1}^{n} b_k = \lim_{n\to\infty} n\left[\Phi\left(I - \Phi\right)^{-1}\omega_1 - \left(I - \Phi\right)^{-1}\delta_1\right] - \ldots \\ \ldots - \Phi^2\left(I - \Phi\right)^{-2}\omega_1 + \Phi\left(I - \Phi\right)^{-2}\delta_1 \end{aligned} \tag{69}$$

where two last terms in equation (69) is constant and two first terms increase linearly with n. This implies that generally, expression $\sum_{k=1}^{n-1} b_k$ doesn't have a limit as $n \to \infty$. The relation (69) is, however, useful to determine restrictions on parameters that determine limiting the dynamics of a_n.

Finally, we analyze the limiting behaviour of the term $\sum_{k=1}^{n-1} b_k'\Sigma\Sigma' b_k$ that appears in the equation (67). First we note that

$$\sum_{k=1}^{n-1} b_k'\Sigma\Sigma' b_k = \sum_{k=1}^{n-1}\left(\Sigma' b_k\right)'\Sigma' b_k \tag{70}$$

where, following equation (64), we have

$$\Sigma' b_n = \Sigma'\left[\Phi\left(I - \Phi^n\right)\left(I - \Phi\right)^{-1}\omega_1 - \left(I - \Phi^n\right)\left(I - \Phi\right)^{-1}\delta_1\right] \tag{71}$$

$$\left(\Sigma' b_n\right)' = \left[\Phi\left(I - \Phi^n\right)\left(I - \Phi\right)^{-1}\omega_1 - \left(I - \Phi^n\right)\left(I - \Phi\right)^{-1}\delta_1\right]'\Sigma \tag{72}$$

[22] E.g., see Hamilton (1994).

and therefore

$$\left(\Sigma' b_n\right)' \Sigma' b_n = \left[\Phi\left(I-\Phi^n\right)\left(I-\Phi\right)^{-1}\omega_1 - \left(I-\Phi^n\right)\left(I-\Phi\right)^{-1}\delta_1\right]' \Sigma\times \\ \times\,\Sigma'\left[\Phi\left(I-\Phi^n\right)\left(I-\Phi\right)^{-1}\omega_1 - \left(I-\Phi^n\right)\left(I-\Phi\right)^{-1}\delta_1\right] \tag{73}$$

which implies that

$$\sum_{k=1}^{n}\left(\Sigma' b_k\right)'\Sigma' b_k = \underbrace{\sum_{k=1}^{n}\left[\Phi\left(I-\Phi^k\right)\left(I-\Phi\right)^{-1}\omega_1\right]'\Sigma\Sigma'\Phi\left(I-\Phi^k\right)\left(I-\Phi\right)^{-1}\omega_1 +}_{1} \\ + \underbrace{\sum_{k=1}^{n}\left[\left(I-\Phi^k\right)\left(I-\Phi\right)^{-1}\delta_1\right]'\Sigma\Sigma'\left(I-\Phi^k\right)\left(I-\Phi\right)^{-1}\delta_1 -}_{2} \\ -2\underbrace{\sum_{k=1}^{n}\left[\Phi\left(I-\Phi^k\right)\left(I-\Phi\right)^{-1}\omega_1\right]'\Sigma\Sigma'\left(I-\Phi^k\right)\left(I-\Phi\right)^{-1}\delta_1}_{3} \tag{74}$$

Since we're first and foremost interested in the limiting behaviour of pricing parameters, in what follows we will only focus on terms that growth linearly with maturity n (and we will denote these terms with a subscript n for clarity). In fact, in the limit, terms of (74) either growth linearly with n or converge to constant values (under the assumption 65). We, therefore, focus on former terms, which characterize the limiting behaviour of a_n and are necessary to derive the constraint that guarantees convergence of prices of dividend strips to zero.

$$\begin{aligned} 1_{n\to\infty} &= n\left[\left(\Phi\left(I-\Phi\right)^{-1}\omega_1\right)'\Sigma\Sigma'\Phi\left(I-\Phi\right)^{-1}\omega_1\right] \\ 2_{n\to\infty} &= n\left[\left(\left(I-\Phi\right)^{-1}\delta_1\right)'\Sigma\Sigma'\left(I-\Phi\right)^{-1}\delta_1\right] \\ 3_{n\to\infty} &= n\left[\left(\Phi\left(I-\Phi\right)^{-1}\omega_1\right)'\Sigma\Sigma'\left(I-\Phi\right)^{-1}\delta_1\right] \end{aligned} \tag{75}$$

which implies that

$$\left(\sum_{k=1}^{n}\left(\Sigma' b_k\right)'\Sigma' b_k\right)_{n\to\infty} = 1_{n\to\infty} + 2_{n\to\infty} - 2\cdot 3_{n\to\infty} \tag{76}$$

This allows us to analyze the limiting properties of a_n as $n\to\infty$

$$a_{n\to\infty} = n\omega_0 + n\omega_1'\mu^Q + n\left[\Phi\left(I-\Phi\right)^{-1}\omega_1 - \left(I-\Phi\right)^{-1}\delta_1\right]'\mu^Q + \frac{n}{2}\omega_1'\Sigma\Sigma'\omega_1 + \dots \\ \dots + n\omega_1'\Sigma\Sigma'\left[\Phi\left(I-\Phi\right)^{-1}\omega_1 - \left(I-\Phi\right)^{-1}\delta_1\right] + \frac{1}{2}\left[1_{n\to\infty} + 2_{n\to\infty} - 2\cdot 3_{n\to\infty}\right] - n\delta_0 \tag{77}$$

We now recall that deterministic maturity-varying parameters a_n and b_n are used to price dividend strips via equation (50). In order to ensure that prices of dividend strips converge to zero as time to maturity n approaches infinity, we impose the following restriction on the model parameters

$$\begin{aligned}
C = \omega_0 + \omega_1' \mu^Q + \left[\Phi \left(I-\Phi\right)^{-1} \omega_1 - \left(I-\Phi\right)^{-1} \delta_1\right]' \mu^Q + \frac{1}{2} \omega_1' \Sigma \Sigma' \omega_1 + \dots \\
\dots + \omega_1' \Sigma \Sigma' \left[\Phi \left(I-\Phi\right)^{-1} \omega_1 - \left(I-\Phi\right)^{-1} \delta_1\right] + \dots \\
\dots + \frac{1}{2} \left[\left(\Phi \left(I-\Phi\right)^{-1} \omega_1\right)' \Sigma \Sigma' \Phi \left(I-\Phi\right)^{-1} \omega_1\right] + \dots \\
\dots + \frac{1}{2} \left[\left(\left(I-\Phi\right)^{-1} \delta_1\right)' \Sigma \Sigma' \left(I-\Phi\right)^{-1} \delta_1\right] - \dots \\
\dots - \left[\left(\Phi \left(I-\Phi\right)^{-1} \omega_1\right)' \Sigma \Sigma' \left(I-\Phi\right)^{-1} \delta_1\right] - \delta_0 < 0
\end{aligned} \tag{78}$$

which implies that in the limit

$$\lim_{n \to \infty} a_n = -\infty \tag{79}$$

and since b_n and X_t are finite, as seen from equation (66), this implies for any time t the price of a (normalized) dividend strip satisfies

$$\lim_{n \to \infty} V_{t,n}^d = 0 \tag{80}$$

The price of equity (normalized by current dividend) is an infinite sum of individual dividend strips and is given by equation (37)

$$\frac{V_t}{D_t} = \sum_{n=1}^{\infty} \exp\left(a_n + b_n' X_t\right) = \sum_{n=1}^{\infty} V_{t,n}^d \tag{81}$$

Finally, we note that condition (78) is also sufficient for equity-dividend ratio (81) to converge.

D Tables and figures

Table 8: Correlations between the macro factors and the underlying time series over the 1990-2017 period.

Correlations with factor 1	
Correlation	Dataseries
74%	'MNI Chicago Business Barometer (sa)'
74%	'Eurostat Order Book Level'
74%	'S&P GSCI Industrial Metals Index Spt'
74%	'US PMI Business Imports'
74%	'STCA Canada Labor Force Unemployment Rate SA'
75%	'Capacity utilization rate'
75%	'Bureau of National Affairs US Wage Trend Indicator'
75%	'European Commission Capacity Utilization UK SA'
76%	'European Commission Capacity Utilization UK SA'
77%	'European Commission Manufacturing Confidence UK Industrial Confidence Indicator'
77%	'STCA Canada Labor Force Unemployment SA'
77%	'US PMI'
77%	'ISM Manufacturing PMI SA'
77%	'US PMI'
77%	'ISM Manufacturing PMI SA'
79%	'Capacity utilization rate Switzerland'
79%	'Unemployment Rate'
80%	'Capacity utilization rate Italy'
81%	'Capacity utilization rate Germany'
85%	'ISM Employment'
Correlations with factor 2	
Correlation	Dataseries
55%	'Service Sales Price Expectations France'
56%	'Retail Sales Price Expectations France'
57%	'GSCI Energy'
57%	'GSCI Energy'
57%	'Import Price Index Switzerland'
58%	'OECD Inflation'
58%	'GSCI Agriculture'
58%	'GSCI Agriculture'
58%	'OECD Inflation'
58%	'OECD Inflation'
60%	'OECD Inflation'
-60%	'OECD Leading Global'
60%	'Retail Sales Price Expectations Germany'
68%	'US Producer Price Index All Commodities'
68%	'Import price index ex petroleum'
69%	'Production and Import Price Index Switzerland'
71%	'US PPI'
73%	' Producer Price Index Domestic Demand Products'
76%	'Producer Price Japan'
79%	'Import prices'
Correlations with factor 3	
Correlation	Dataseries
-40%	'European Commission Retail Trade Confidence UK Employment Expectations'
-40%	'European Commission Retail Trade Confidence UK Employment Expectations'
-43%	'Canada Employment Ratio'
-43%	'France Unemployment Rate '
-43%	'France Expected Business Conditions'
44%	'Trade Weighted Dollar'
44%	'UK saving rate'
-47%	'Economic sentiment EMU'
47%	'US PMI'
-47%	'Economic sentiment France'
-47%	'EC Consumer Confidence Eurozone Financial Situation of Households Next 12 Month'
-49%	'EC Consumer Confidence Eurozone General Economic Situation Next 12 Month'
-49%	'NFIB Expected Credit Conditions'
-50%	'Economic sentiment Italy'
-50%	'EC Consumer Confidence Eurozone Major Purchases Next 12 Months'
50%	'Gold price'
-51%	'Canada unemployment rate'
-51%	'EC Consumer Confidence UK Unemployment Expectations Over The Next 12 Month'
-51%	'EC Consumer Confidence Italy General Economic Situation Next 12 Month'
-54%	'UK Unemployment ILO Unemployment Rate SA'

Table 9: List of the dataseries used to build the macro factors. Period: 1990-2017.

Data number	Country	Data series
1	Australia	Australia leading index
2	Australia	IMF DOT Australia Imports from Japan
3	Brazil	Anfavea Brazil Vehicle Production
4	Brazil	OECD Brazil Bus. Tend. Manuf.Capacity Util. EC & Natl Ind. StckSA %
5	Brazil	OECD Brazil Prod. Manufacturing Total Manufacturing GrthSA %
6	Canada	Canada Employment Ratio
7	Canada	Canada Housing Mortgage Co Housing Starts Urban Areas Multiple SAAR
8	Canada	Canada Housing Mortgage Co Housing Starts Urban Areas Single SAAR
9	Canada	Canada Mortgage and Housing Corp Total Starts SAAR
10	Canada	Canada New Motor Vehicle Sales Passenger Cars Country of Manufacture NSA Unit
11	Canada	Canada New Motor Vehicle Sales Total Country of Manufacture NSA Unit
12	Canada	Canada New Motor Vehicle Sales Trucks Country of Manufacture NSA Unit
13	Canada	Canada Total Dwellings Units
14	Canada	Canada unemployment rate
15	Canada	Industrial production costs
16	Canada	STCA Canada Labor Force Unemployment Rate SA
17	Canada	STCA Canada Labor Force Unemployment SA
18	Canada	STCA Canada Monthly Survey of Manufacturing New Orders SA CAD
19	Canada	STCA Canada Monthly Survey of Manufacturing Sales SA CAD
20	Canada	STCA Canada Net Change in Labor Force Employment SA
21	China	Past Building Activities
22	China	Residential property prices
23	Eurozone	Capacity utilization rate EMU
24	Eurozone	EC Construction Confidence Eurozone Employment Expectations
25	Eurozone	EC Consumer Confidence Eurozone Financial Situation of Households Next 12 Month
26	Eurozone	EC Consumer Confidence Eurozone General Economic Situation Next 12 Month
27	Eurozone	EC Consumer Confidence Eurozone Major Purchases Next 12 Months
28	Eurozone	EC Eurozone Expected Unemployment
29	Eurozone	Economic sentiment EMU
30	Eurozone	Euro Area Manufacturing Confidence
31	Eurozone	Euro Trade Weighted
32	Eurozone	European Commission Capacity Utilization UK SA
33	Eurozone	European Commission Economic SentiMent Indicator UK
34	Eurozone	European Commission Manufacturing Confidence
35	Eurozone	European Commission Manufacturing Confidence Eurozone Employment Expectations
36	Eurozone	European Commission Manufacturing Confidence Eurozone Industrial Confidence
37	Eurozone	European Commission Retail Confidence Eurozone Intentions of Placing Orders
38	Eurozone	European Commission Retail Trade Confidence Eurozone Employment Expectations
39	Eurozone	Eurostat Expected Production
40	Eurozone	Eurostat Exports Orders
41	Eurozone	Eurostat Manufacturing Survey
42	Eurozone	Eurostat Order Book Level
43	Eurozone	Expected housing price
44	Eurozone	M3 Money growth MA
45	Eurozone	Retail Sales Price Expectations EMU
46	France	EC Consumer Confidence France General Economic Situation Next 12 Month
47	France	Economic sentiment France
48	France	France Expected Business Conditions
49	France	France Unemployment Rate
50	France	France Wages and Salaries Current Prices
51	France	Retail Sales Price Expectations France
52	France	Service Sales Price Expectations France
53	Germany	Capacity utilization rate Germany
54	Germany	EC Consumer Confidence Germany General Economic Situation Next 12 Month
55	Germany	Economic sentiment Germany
56	Germany	Germany Import Price Index
57	Germany	Germany Manufacturing Confidence
58	Germany	Retail Sales Price Expectations Germany
59	India	INR Currency
60	Italy	Capacity utilization rate Italy
61	Italy	EC Consumer Confidence Italy General Economic Situation Next 12 Month
62	Italy	Economic sentiment Italy
63	Italy	Italy Unemployment Rate
64	Italy	Retail Sales Price Expectations Italy

Table 10: Continued list of the dataseries used to build the macro factors. Period: 1990-2017.

Data number	Country	Data series
65	Japan	IMF DOT Japan Exports to Europe FOB
66	Japan	Import prices
67	Japan	Japan Bankruptcies Total Debt YoY
68	Japan	Japan Capacity Utilization Operating Ratio Iron & Steel SA
69	Japan	Japan Capacity Utilization Operating Ratio Manufacturing SA
70	Japan	Japan Domestically Licensed Banks New Housing Loans
71	Japan	Japan Housing Starts YoY NSA
72	Japan	Japan Labor Force Employed SA
73	Japan	Japan leading index
74	Japan	Japan Merchandise Trade Exports YoY NSA
75	Japan	Japan Money Stock Broadly-defined Lqdt Avg amts outstanding YoY%
76	Japan	Japan New Composite Index of Business Cycle Indicators Leading Index
77	Japan	Japan New Diffusion Index of Business Cycle Indicators Leading Index
78	Japan	Japan Real Exports MoM%
79	Japan	Japan Retail Sales General Merchandise
80	Japan	Japan Tankan Fixed Investments Large Enterprises Manufacturing NLA Method
81	Japan	Japan Unemployment Rate
82	Japan	Japan Unemployment Rate SA
83	Japan	Japanese cars production
84	Japan	OECD Japan Working Hours
85	Japan	Producer Price Japan
86	Japan	Tokyo Condominium Sales YoY
87	Japan	Tokyo Stock Exchange Tokyo Price Index TOPIX
88	Mexico	Goods exports Mexico
89	Mexico	Mexican Vehicle Sales Auto+truck NSA
90	Mexico	Mexico Seasonally Adjusted Coincident Indicator
91	Mexico	Mexico Seasonally Adjusted Leading Indicator
92	Mexico	Mexico Vehicle Production Total Production
93	Mexico	Nomial value of Mexican exports
94	None	Bloomberg Financial Conditions
95	None	Gold price
96	None	GSCI Agriculture
97	None	GSCI Energy
98	None	GSCI Industrial Metals
99	None	Oil prices
100	None	S&P GSCI Agricultural Index Spot CME
101	None	S&P GSCI Energy Index Spot CME
102	None	S&P GSCI Industrial Metals Index Spt
103	OECD	OECD Car Sales
104	OECD	OECD Inflation
105	OECD	OECD Leading Global
106	OECD	OECD Mexcio Leading
107	OECD	OECD Total CPI All Items Total 2010=100
108	South Africa	Exports ex Gold
109	South Africa	Private sector credit
110	South Africa	South Africa Consumer Confidence Financial Position During Next 12m.
111	South Africa	South Africa Consumer Confidence Rating of Present Time To Buy Durables
112	South Africa	South Africa Current Account SA - Less Merchandise Imports
113	Switzerland	Capacity utilization rate Switzerland
114	Switzerland	IMF Swiss Banks Assets
115	Switzerland	Import Price Index Switzerland
116	Switzerland	Personal Income
117	Switzerland	Personal Income Index
118	Switzerland	Personal Income Real Index
119	Switzerland	Production and Import Price Index Switzerland
120	Switzerland	Swiss equities
121	Switzerland	Swiss Franc Effective Exchange Rate
122	Switzerland	Switzerland mortgage loans
123	Switzerland	UBS housing bubble index

Table 11: Continued list of the dataseries used to build the macro factors. Period: 1990-2017.

Data number	Country	Data series
124	UK	EC Construction Confidence UK Employment Expectations
125	UK	EC Consumer Confidence UK Unemployment Expectations Over The Next 12 Month
126	UK	European Commission Manufacturing Confidence UK Employment Expectations
127	UK	European Commission Manufacturing Confidence UK Industrial Confidence Indicator
128	UK	European Commission Retail Trade Confidence UK Employment Expectations
129	UK	GFK UK Consumer Confidence Indicator
130	UK	UK CBI MTE Full Volume of Output Next 3 Months Balance
131	UK	UK CBI MTE Full Volume of Total Order Book Balance
132	UK	UK Claimant Count Rate SA
133	UK	UK Retail Sales All Retailing Sales Ex Automotive Fuel Chained Volume SA
134	UK	UK saving rate
135	UK	UK Unemployment ILO Unemployment Rate SA
136	US	Bureau of National Affairs US Wage Trend Indicator
137	US	Capacity utilization rate
138	US	Cleveland Fed Expected 5Y Inflation
139	US	Composite Business Cycle Indicator - Coincident Indicator
140	US	Composite Business Cycle Indicator - Leading Indicator
141	US	Composite Business Cycle Indicator - Leading Indicator YoY
142	US	Conference Board Consumer Confidence SA 1985=100
143	US	Conference Board US Diffusion index new orders
144	US	Conference Board US Leading Index MoM
145	US	Construction order book evolution
146	US	Dallas Federal Reserve Japan Real Personal Disposable Income Index
147	US	Dallas Federal Reserve South Africa International Real House Price Index
148	US	Dollar index
149	US	Expected Wages Over the Next 6 months
150	US	Federal Reserve Consumer Credit Total Net Change SA
151	US	Import price index ex petroleum
152	US	ISM Employment
153	US	ISM Manufacturing
154	US	ISM Manufacturing PMI SA
155	US	ISM New Orders
156	US	Jobless claims
157	US	Michigan Expected Inflation
158	US	MNI Chicago Business Barometer (sa)
159	US	Mortgage rate
160	US	NABE US Industry Demand Survey Wages and Salaries
161	US	National Association of Home Builders Market Index SA
162	US	NFIB Expected Credit Conditions
163	US	NFIB Good Time to Expand
164	US	NFIB Hiring Plans
165	US	Philadelphia Fed Business Outlook Survey Diffusion Index General Conditions
166	US	Private Housing Authorized by Bldg Permits by Type Total SAAR
167	US	Saving Rate
168	US	Trade Weighted Dollar
169	US	U-3 US Unemployment Rate Total in Labor Force Seasonally Adjusted
170	US	Unemployment Rate
171	US	US Capacity Utilization % of Total Capacity SA
172	US	US Census Bureau US Construction Spending
173	US	US Cleveland Fed Expected 10Y Inflation
174	US	US Cleveland Fed Expected 1Y Inflation
175	US	US Employees on Nonfarm Payrolls Manufacturing Industry Monthly Net Change SA
176	US	US Employees on Nonfarm Payrolls Total MoM Net Change SA
177	US	US Indexes Imports Of Goods & Services
178	US	US Initial Jobless Claims SA
179	US	US ISM
180	US	US New One Family Houses Sold Annual Total SAAR
181	US	US New Privately Owned Housing Units Started by Structure Total SAAR
182	US	US PMI
183	US	US PMI Business Imports
184	US	US PPI
185	US	US Producer Price Index All Commodities
186	US	US Producer Price Index Domestic Demand Products

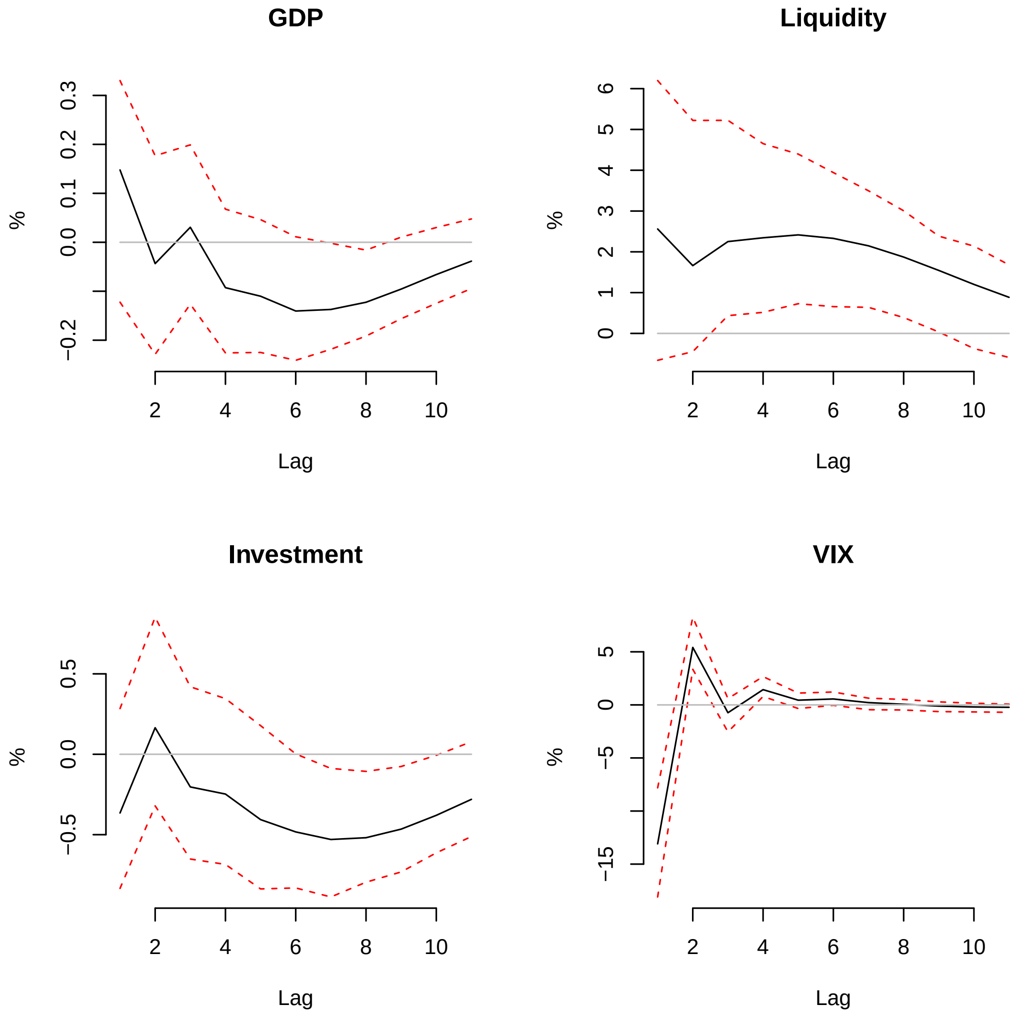

Figure 8: Impulse responses of US GDP, Liquidity, Investment and VIX to the bubble in Dow Jones equity index.

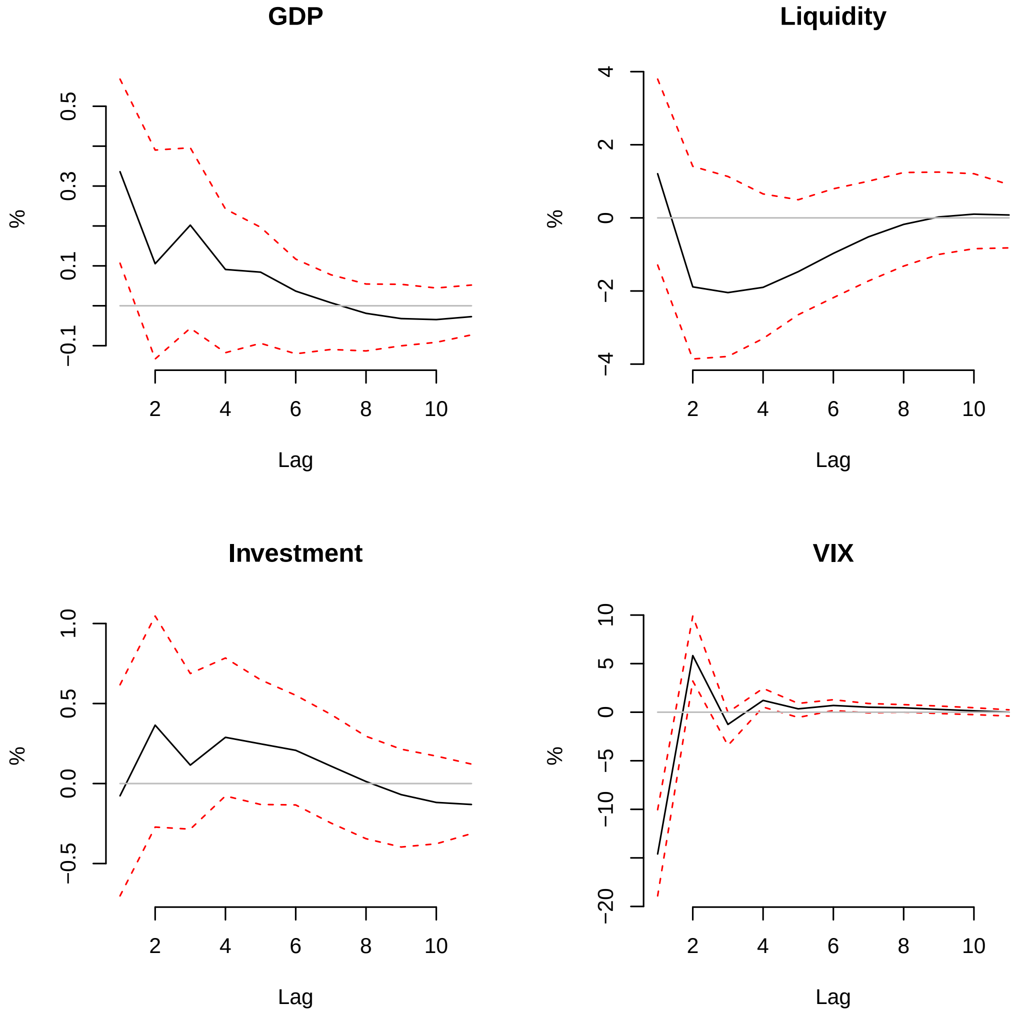

Figure 9: Impulse responses of US GDP, Liquidity, Investment and VIX to the bubble in Russell equity index.

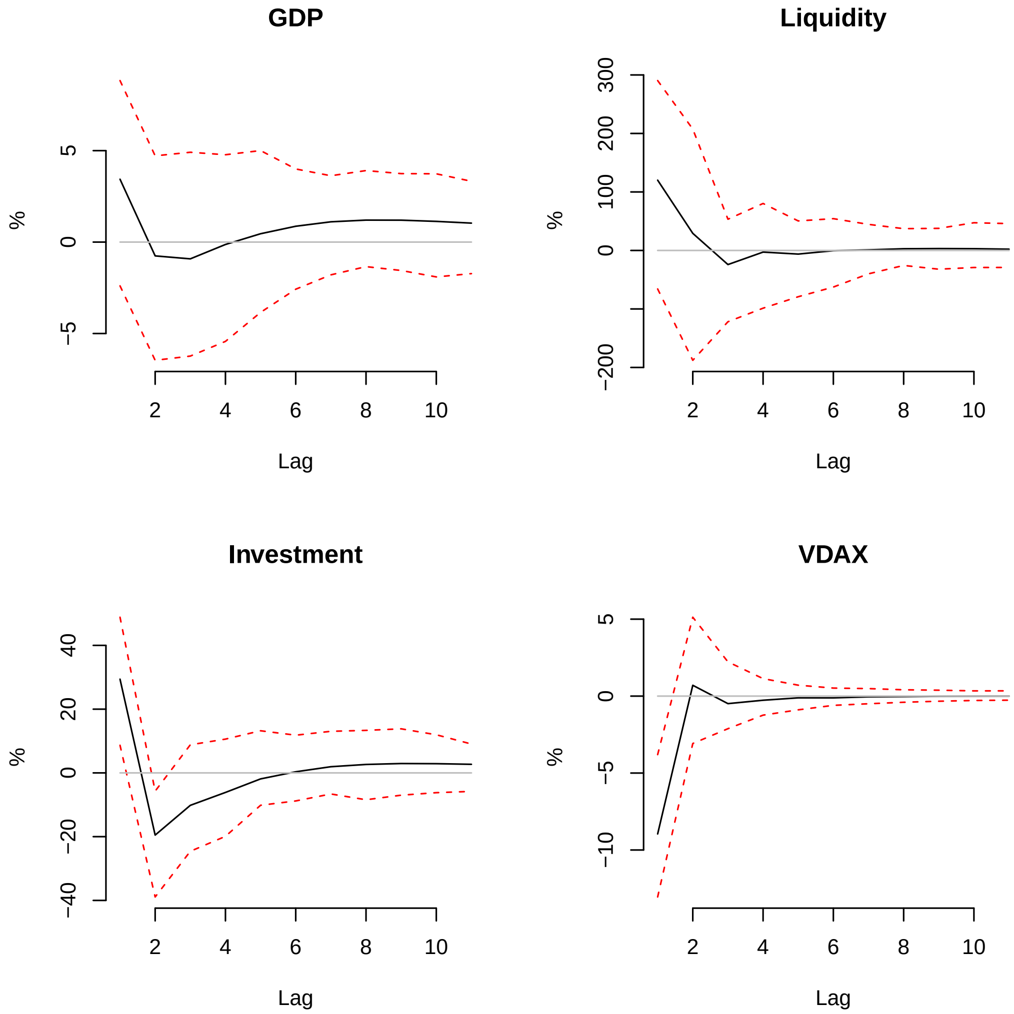

Figure 10: Impulse responses of Europe GDP, Liquidity, Investment and VIX to the bubble in MSCI Europe equity index.

References

Alessi, L., Barigozzi, M., and Capasso, M. (2010). Improved penalization for determining the number of factors in approximate factor models. *Statistics & Probability Letters*, 80(23):1806–1813.

Ang, A. and Liu, J. (2001). A general affine earnings valuation model. *Review of Accounting Studies*, 6(4):397–425.

Ang, A. and Piazzesi, M. (2003). A no-arbitrage vector autoregression of term structure dynamics and macroeconomic and latent variables. *Journal of Monetary Economics*, 50:745–787.

Ang, A., Piazzesi, M., and Wei, M. (2006). What does the yield curve tell us about gdp growth? *Journal of econometrics*, 131(1):359–403.

Ang, A. and Ulrich, M. (2012). Nominal bonds, real bonds, and equity. *Netspar Discussion Paper No. 12/2011-103*.

Bansal, R. and Yaron, A. (2004). Risks for the long run: A potential resolution of asset pricing puzzles. *The Journal of Finance*, 59(4):1481–1509.

Bauer, G. H. and Diez de los Rios, A. (2012). An international dynamic term structure model with economic restrictions and unspanned risks. Technical report, Bank of Canada working paper.

Bauer, M. D. and Rudebusch, G. D. (2016). Resolving the spanning puzzle in macro-finance term structure models. *Review of Finance*, 21(2):511–553.

Beber, A., Brandt, M. W., and Luisi, M. (2015). Distilling the macroeconomic news flow. *Journal of Financial Economics*, 117(3):489–507.

Bekaert, G. and Grenadier, S. (1999). Stock and bond pricing in an affine economy. *NBER*.

Belo, F., Collin-Dufresne, P., and Goldstein, R. S. (2015). Dividend dynamics and the term structure of dividend strips. *The Journal of Finance*, 70(3):1115–1160.

Bernanke, B. S., Boivin, J., and Eliasz, P. (2005). Measuring the effects of monetary policy: a factor-augmented vector autoregressive (favar) approach. *The Quarterly journal of economics*, 120(1):387–422.

Brunnermeier, M. K. (2010). Bubbles, liquidity, and the macroeconomy. *NBER Reporter Online*, 01(2):10–12.

Chen, L., Da, Z., and Priestley, R. (2012). Dividend smoothing and predictability. *Management science*, 58(10):1834–1853.

Cheridito, P., Filipović, D., and Kimmel, R. L. (2007). Market price of risk specifications for affine models: Theory and evidence. *Journal of Financial Economics*, 83(1):123–170.

Christensen, J. H., Diebold, F. X., and Rudebusch, G. D. (2011). The affine arbitrage-free class of nelson–siegel term structure models. *Journal of Econometrics*, 164(1):4–20.

Cochrane, J. H. and Piazzesi, M. (2009). Decomposing the yield curve. *Working Paper*.

Collin-Dufresne, P., Goldstein, R. S., and Jones, C. S. (2008). Identification of maximal affine term structure models. *The Journal of Finance*, 63(2):743–795.

Constantinides, G. M. (1992). A theory of the nominal term structure of interest rates. *Review of Financial Studies*, 5(4):531–552.

Dai, Q. and Singleton, K. J. (2000). Specification analysis of affine term structure models. *The Journal of Finance*, 55(5):1943–1978.

Dai, Q. and Singleton, K. J. (2002). Expectation puzzles, time-varying risk premia, and affine models of the term structure. *Journal of financial Economics*, 63(3):415–441.

Duffee, G. R. (2002). Term premia and interest rate forecasts in affine models. *The Journal of Finance*, 57(1):405–443.

Duffie, D. (2004). *Credit risk modeling with affine processes*. Scuola normale superiore.

Duffie, D., Filipović, D., and Schachermayer, W. (2003). Affine processes and applications in finance. *Annals of applied probability*, pages 984–1053.

Duffie, D. and Kan, R. (1996). A yield-factor model of interest rates. *Mathematical finance*, 6(4):379–406.

Eraker, B. (2008). Affine general equilibrium models. *Management Science*, 54(12):2068–2080.

Estrella, A. and Hardouvelis, G. A. (1991). The term structure as a predictor of real economic activity. *The journal of Finance*, 46(2):555–576.

Evans, G. W. (1991). Pitfalls in testing for explosive bubbles in asset prices. *The American Economic Review*, 81(4):922–930.

Faust, J., Rogers, J. H., and Wright, J. H. (2005). News and noise in g-7 gdp announcements. *Journal of Money, Credit, and Banking*, 37(3):403–419.

Fisher, M. and Gilles, C. (1998). Around and around: The expectations hypothesis. *The Journal of Finance*, 53(1):365–383.

Flannery, M. J. and Protopapadakis, A. A. (2002). Macroeconomic factors do influence aggregate stock returns. *The review of financial studies*, 15(3):751–782.

Ghysels, E., Horan, C., and Moench, E. (2014). Forecasting through the rear-view mirror: Data revisions and bond return predictability. *NY Fed Staff Reports*.

Grossman, H. and Diba, B. (1988). Explosive rational bubbles in stock prices? *American Economic Review*, 78:520–30.

Gürkaynak, R. S. (2008). Econometric tests of asset price bubbles: taking stock. *Journal of Economic surveys*, 22(1):166–186.

Hall, S. G., Psaradakis, Z., and Sola, M. (1999). Detecting periodically collapsing bubbles: a markov-switching unit root test. *Journal of Applied Econometrics*, pages 143–154.

Hamilton, J. D. (1994). *Time series analysis*, volume 2. Princeton university press Princeton.

Hamilton, J. D. and Wu, J. C. (2012). Identification and estimation of gaussian affine term structure models. *Journal of Econometrics*, 168(2):315–331.

Harrison, J. M. and Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic theory*, 20(3):381–408.

Harvey, C. R. (1988). The real term structure and consumption growth. *Journal of Financial Economics*, 22(2):305–333.

Hirano, T. and Yanagawa, N. (2016). Asset bubbles, endogenous growth, and financial frictions. *The Review of Economic Studies*, 84(1):406–443.

Hördahl, P., Tristani, O., and Vestin, D. (2006). A joint econometric model of macroeconomic and term-structure dynamics. *Journal of Econometrics*, 131(1):405–444.

Jardet, C., Monfort, A., and Pegoraro, F. (2013). No-arbitrage near-cointegrated var (p) term structure models, term premia and gdp growth. *Journal of Banking & Finance*, 37(2):389–402.

Joslin, S., Priebsch, M., and Singleton, K. J. (2014). Risk premiums in dynamic term structure models with unspanned macro risks. *The Journal of Finance*, 69(3):1197–1233.

Joslin, S., Singleton, K. J., and Zhu, H. (2011). A new perspective on gaussian dynamic term structure models. *The Review of Financial Studies*, 24(3):926–970.

Kim, D. H. (2007). Challenges in macro-finance modeling. *Working Paper*.

Kim, D. H. and Wright, J. H. (2005). An arbitrage-free three-factor term structure model and the recent behavior of long-term yields and distant-horizon forward rates. *Working Paper*.

Leary, M. T. and Michaely, R. (2011). Determinants of dividend smoothing: Empirical evidence. *The Review of Financial Studies*, 24(10):3197–3249.

Lemke, W. and Werner, T. (2009). The term structure of equity premia in an affine arbitrage-free model of bond and stock market dynamics. *European Central Bank*.

LeRoy, S. F. and Porter, R. D. (1981). The present-value relation: Tests based on implied variance bounds. *Econometrica: Journal of the Econometric Society*, pages 555–574.

Lettau, M. and Wachter, J. (2011). The term structures of equity and interest rates. *Journal of Financial Economics*, 101:90–113.

Martin, A. and Ventura, J. (2012). Economic growth with bubbles. *The American Economic Review*, 102(6):3033–3058.

Mönch, E. (2008a). Forecasting the yield curve in a data-rich environment: A no-arbitrage factor-augmented var approach. *Journal of Econometrics*, 146(1):26–43.

Mönch, E. (2008b). Forecasting the yield curve in a data-rich environment: A no-arbitrage factor-augmented var approach. *Journal of Econometrics*, 146(1):26–43.

Monfort, A. and Pegoraro, F. (2007). Switching varma term structure models-extended version. *Working Paper*.

Piazzesi, M. (2010). Affine term structure models. *Handbook of financial econometrics*, 1:691–766.

Rudebusch, G. D. and Wu, T. (2008). A macro-finance model of the term structure, monetary policy and the economy. *The Economic Journal*, 118(530):906–926.

Shiller, R. (1981). Do stock prices move too much to be justified by subsequent changes in dividends? *American Economic Review*, 71:421–436.

Simon, D. P. (2003). The nasdaq volatility index during and after the bubble. *The Journal of Derivatives*, 11(2):9–24.

Sornette, D., Woodard, R., and Zhou, W.-X. (2009). The 2006–2008 oil bubble: Evidence of speculation, and prediction. *Physica A: Statistical Mechanics and its Applications*, 388(8):1571–1576.

Swanson, E. et al. (2014). A macroeconomic model of equities and real, nominal, and defaultable debt. *Unpublished Manuscript, University of California, Irvine*.

Trapani, L. (2017). A randomised sequential procedure to determine the number of factors. *Forthcoming in the Journal of the American Statistical Association*.

Van Binsbergen, J., Brandt, M., and Koijen, R. (2012). On the timing and pricing of dividends. *The American Economic Review*, 102(4):1596–1618.

Van Binsbergen, J. H. and Koijen, R. S. (2017). The term structure of returns: Facts and theory. *Journal of Financial Economics*, 124(1):1–21.

West, K. D. (1988). Bubbles, fads and stock price volatility tests: a partial evaluation. *The Journal of Finance*, 43(3):639–656.

Wu, T. (2001). Macro factors and the affine term structure of interest rates. *Working Paper*.